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Q1) $u = \log(x^3+y^3-x^2y-xy^2)$

i) To prove: $x\frac{du}{dx} + y\frac{du}{dy} = 3$

$\therefore \frac{du}{dx} = \frac{1}{x^3+y^3-x^2y-xy^2}(3x^2-2yx-y^2)$

$\frac{x\,du}{dx} = \frac{3x^3-2yx^2-y^2x}{x^3+y^3-x^2y-xy^2} \rightarrow (1)$

$\frac{du}{dy} = \frac{1}{x^3+y^3-x^2y-xy^2}(3y^2-x^2-2xy)$

$\frac{y\,du}{dy} = \frac{3y^3-x^2y-2xy^2}{x^3+y^3-x^2y-xy^2} \rightarrow (2)$

eq (1) + (2)

$\frac{x\,du}{dy} + \frac{y\,du}{dy} = \frac{3x^3-2yx^2-y^2x+3y^3-x^2y-2xy^2}{x^3+y^3-x^2y-xy^2}$

$= \frac{3x^3+3y^3-3yx^2-3xy^2}{x^3+y^3-x^2y-xy^2}$

$= \frac{3(x^3+y^3-yx^2-xy^2)}{x^3+y^3-x^2y-xy^2} = 3$

Hence Proved.

ii)

$$u = \log(x^3 + y^3 - x^2y - xy^2)$$

$$= \log(x^3 - x^2y + y^3 - xy^2)$$

$$= \log((x^3 - x^2y) + y^3 - xy^2)$$

$$= \log(x^2(x-y) + y^2(y-x))$$

$$= \log(x^2(x-y) \cancel{+} - y^2(x-y))$$

$$= \log((x^2-y^2)(x+y))$$

$$u = \log((x-y)^2(x+y))$$

$$u = \log(x+y) + \log(x-y)^2$$

$$u = \log(x+y) + 2\log(x-y)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x+y} + \frac{2}{x-y}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-1}{(x+y)^2} - \frac{2}{(x-y)^2} \longrightarrow 1$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+y} - \frac{2}{x-y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-1}{(x+y)^2} - \frac{2}{(x-y)^2} \rightarrow 2$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{(x+y)^2} + \frac{2}{(x-y)^2} \rightarrow 3$$

From ①, ② & ③

Adding eq ① + ② + 2③

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\partial^2 u}{\partial x \partial y}$$

$$= \frac{-1}{(x+y)^2} - \frac{2}{(x-y)^2} + \left(\frac{-1}{(x+y)^2} - \frac{2}{(x-y)^2}\right)$$

$$+ 2\left(\frac{-1}{(x+y)^2} + \frac{2}{(x-y)^2}\right)$$

$$= \frac{-1}{(x+y)^2} - \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} - \frac{2}{(x-y)^2} - \frac{2}{(x+y)^2}$$

$$+ \frac{4}{(x-y)^2}$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial x^2} + 2\cdot\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-4}{(x+y)^2}}$$

Q2) $u = \frac{1}{r}$, $r = \sqrt{x^2+y^2+z^2}$

To prove: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore \frac{\partial u}{\partial x} = \frac{du}{dr} \times \frac{\partial r}{\partial x} = \frac{-1}{r^2} \times \frac{x}{r} = \frac{-x}{r^3}$

Hence $\frac{\partial^2 u}{\partial x^2} = \frac{-1}{r^3} + \frac{3x}{r^4} \cdot \frac{\partial r}{\partial x} = \frac{-1}{r^3} + \frac{3x^2}{r^5} \rightarrow (1)$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{-1}{r^3} + \frac{3y^2}{r^5} \rightarrow (2)$

& $\frac{\partial^2 u}{\partial z^2} = \frac{-1}{r^3} + \frac{3z^2}{r^5} \rightarrow (3)$

Adding (1), (2) & (3)

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{r^3} + \frac{3(x^3+y^2+z^2)}{r^5}$

$= -\frac{3}{r^3} + \frac{3r^2}{r^5}$ (From given)

$= -\frac{3}{r^3} + \frac{3}{r^3}$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Hence Proved

Q3) To Prove : $\frac{du}{dt} = 4e^{2t}$

$u = x^2+y^2+z^2$ where $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$ (given)

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt} + \frac{\partial u}{\partial z} \times \frac{dz}{dt}$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial u}{\partial z} = 2z$$

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = e^t(\sin t + \cos t), \quad \frac{dz}{dt} = e^t(-\sin t + \cos t)$$

$$\frac{du}{dt} = 2x e^t + 2y(e^t(\sin t + \cos t)) + 2z e^t(-\sin t + \cos t)$$

$$= 2x^2 + 2y^2 + 2z^2 + 2yz - 2yz \quad \text{(From given)}$$

$$= 2(x^2+y^2+z^2)$$

$$\frac{du}{dt} = 2(x^2+y^2+z^2) = 2(e^{2t}(1+\sin^2 t + \cos^2 t))$$

$$\frac{du}{dt} = 2 \times 2 \times e^{2t} = 4e^{2t}$$

$$\therefore \frac{du}{dt} = 4e^{2t}$$

Hence Proved