Corollary-2 & 3

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→ Euler's thm

Corollary 2: If z is homogeneous function of degree n in x and y, and $z = f(u)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

Corollary 3 If z is homogeneous function of degree n in x and y, and $z = f(u)$ then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \quad \text{where} \quad g(u) = n\frac{f(u)}{f'(u)}$$

→ cor 1.

ex :- $u = e^{x^2+y^2}$

Is u homogeneous ? → No.

but $f(u) = \log u = x^2+y^2$ → this is homogeneous with deg $n=2$

cor 2 → $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2\frac{\log u}{1/u} = 2u\log u$

cor 3 → $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$

$g(u) = n\frac{f(u)}{f'(u)} = 2u\log u$

$= 2u\log u[(2\log u + 2) - 1]$

1. If $u = \sin^{-1}(x\,y\,z)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3\tan u$.

Soln :- u is not homogeneous

but $f(u) = \sin u = xyz$ which is homogeneous with deg $n = 3$

$\therefore$ By cor-2

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)} = 3\frac{\sin u}{\cos u} = 3\tan u$

2. If $u = e^{x^2 f(y/x)}$, prove that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$.

Sol^n :- $u = e^{x^2 f(y/x)}$ is not homogeneous

but $f(u) = \log u = x^2 f\left(\frac{y}{x}\right)$ is homogeneous

with deg $n = 2$

$\therefore$ by cor 2,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2\frac{\log u}{1/u} = 2u\log u$$

3. If $u = \log x + \log y$, prove that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$.

Sol^n :- $u = \log x + \log y = \log(xy)$ is not homogeneous

but $f(u) = e^u = xy$ is homogeneous with

degree $n = 2$

$\therefore$ by cor 2

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2\frac{e^u}{e^u} = 2$$

4. If $u = sin^{-1}\left[\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right]$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\tan u$.

Sol^n :- u is not homogeneous

but $f(u) = \sin u = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$ is homogeneous

with deg $n = \frac{1}{20}$

$\therefore$ By cor - 2

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = \frac{1}{20}\frac{\sin u}{\cos u} = \frac{1}{20}\tan u$$

5. If $u = \frac{x^2y^2z^2}{x^2+y^2+z^2} + \cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$ then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Sol^n :- u is not homogeneous.

but $u = v + w$

where $v = \dfrac{x^2y^2z^2}{x^2+y^2+z^2}$ is homogeneous with deg $n_1 = 4$

and $w = \cos^{-1}\left(\dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$ is not homogeneous

but $f(w) = \cos w = \dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$ is homogeneous with deg $n_2 = \dfrac{1}{2}$

Applying Euler's thm to v,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = n_1 v = 4v \quad \text{—— (1)}$$

Applying cor-2 to $f(w)$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = n\frac{f(w)}{f'(w)} = \frac{1}{2}\frac{\cos w}{(-\sin w)}$$

$$= -\frac{1}{2}\cot w \quad \text{—— (2)}$$

Adding (1) & (2)

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) + z\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\right) = 4v - \frac{1}{2}\cot w$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4\frac{x^2y^2z^2}{x^2+y^2+z^2} - \frac{1}{2}\cot\left[\cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)\right]$$

6. If $x = e^u \tan v$, $y = e^u \sec v$, prove that $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 0$

Soln:- $x = e^u \tan v$, $y = e^u \sec v$.

$$\frac{x}{y} = \frac{e^u \tan v}{e^u \sec v} = \sin v \Rightarrow \boxed{v = \sin^{-1}\left(\frac{x}{y}\right)}$$

$$y^2 - x^2 = e^{2u}(\sec^2 v - \tan^2 v) = e^{2u}$$

$$\boxed{u = \frac{1}{2}\log(y^2 - x^2)}$$

$$\boxed{u = \frac{1}{2}\log(y^2 - x^2)}$$

u is not homogeneous, but $f(u) = e^{2u} = y^2 - x^2$

is homogeneous with deg $n = 2$

By corollary - 2

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2 \cdot \frac{e^{2u}}{2e^{2u}} = 1 \quad \text{---} (1)$$

$v = \sin^{-1}\left(\frac{x}{y}\right)$ is homogeneous function with deg $n = 0$

$\therefore$ By Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 0 \quad \text{---} (2)$$

from (1) & (2)

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 1 \times 0 = 0$$

Hence proved.

7. If $u = tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2sin^3 u \cos u$.

Soln: $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ is not homogeneous

but $f(u) = \tan u = \frac{x^2+y^2}{x+y}$ is homogeneous

with deg $n = 1$.

$\therefore$ By cor - 3

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u) - 1\right]$$

where $g(u) = n\dfrac{f(u)}{f'(u)}$

$= 1 \cdot \dfrac{\tan u}{\sec^2 u}$

$= \sin u \cos u$

$g(u) = \frac{1}{2}\sin 2u$

$\therefore x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \frac{1}{2}\sin 2u\left[\cos 2u - 1\right]$

$= \frac{1}{2} \cdot 2\sin u \cos u\left[-2\sin^2 u\right]$

$= -2\sin^3 u \cos u$

$=$ RHS.

8. If $u = sinh^{-1}\left(\frac{x^3+y^3}{x^2+y^2}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\,tanh^3 u.$

Soln:- u is not homogeneous

but $f(u) = \sinh u = \dfrac{x^3+y^3}{x^2+y^2}$ is homogeneous with deg $n=1$

$\therefore$ By cor - 3

LHS $= g(u)\left[g'(u) - 1\right]$

$g(u) = \dfrac{n f(u)}{f'(u)} = 1 \cdot \dfrac{\sinh u}{\cosh u} = \tanh u$

$\therefore$ LHS $= \tanh u\left[\operatorname{sech}^2 u - 1\right] = \tanh u\left[-\tanh^2 u\right]$

$= -\tanh^3 u =$ RHS.

9. If $u = sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that

(i) $2x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = \tan u$ **(ii)** $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}(\tan^3 u - \tan u)$

Soln :- $f(u) = \sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is homogeneous with deg $n = \frac{1}{2}$

$\therefore$ By cor 2

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = \frac{1}{2}\cdot\frac{\sin u}{\cos u} = \frac{1}{2}\tan u$$

$$2x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = \tan u$$

By cor - 3

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u)-1\right]$$

$$= \frac{1}{2}\tan u\left(\frac{1}{2}\sec^2 u - 1\right]$$

$$= \frac{1}{4}\tan u\left[\sec^2 u - 2\right]$$

$$= \frac{1}{4}\tan u\left(\tan^2 u - 1\right]$$

$$= \frac{1}{4}\left[\tan^3 u - \tan u\right]$$

10. If $u = \log(x^3 + y^3 - x^2y - xy^2)$, prove that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3,$ **(ii)** $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -3.$

Soln :- u is not homogeneous

but $f(u) = e^u = x^3 + y^3 - x^2y - xy^2$ is homogeneous deg $n = 3$

by cor - 2

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 3\frac{e^u}{e^u} = 3$$

by cor - 3

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

and $g(u)=3$

$$=3[0-1]$$

$$=-3$$

= Ans.

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11. If $u=tan^{-1}\left[\frac{x^3+y^3}{2x+3y}\right]$, prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$

Sol^n :- u is not a homogeneous function

but $f(u)=\tan u=\frac{x^3+y^3}{2x+3y}$ is homogeneous with deg 2

By cor - 3

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=g(u)[g'(u)-1]$$

$$g(u)=n\frac{f(u)}{f'(u)}=2\frac{\tan u}{\sec^2 u}$$

$$=2\sin u\cos u$$

$$g(u)=\sin 2u$$

$$\therefore g'(u)=2\cos 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 2u(2\cos 2u-1)$$

$$=2\sin 2u\cos 2u-\sin 2u$$

$$=\sin 4u-\sin 2u.$$

$= \sin 4u - \sin 2u$

$= RHS.$

12. If $u = sin^{-1}\sqrt{x^2+y^2}$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Soln :- u is not homogeneous function

but $f(u) = \sin u = \sqrt{x^2+y^2}$ is homogeneous with deg $n=1$.

By cor - 3

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

where $g(u) = n\frac{f(u)}{f'(u)}$

$\therefore g(u) = 1 \cdot \frac{\sin u}{\cos u} = \tan u$

$g'(u) = \sec^2 u.$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan u[\sec^2 u - 1]$$

$$= \tan^3 u.$$

13. If $u = \frac{1}{3}\log\left(\frac{x^3+y^3}{x^2+y^2}\right)$, find the value of **(i)** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. **(ii)** $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Soln :- u is not homogeneous

but $3u = \log\left(\frac{x^3+y^3}{x^2+y^2}\right)$

$f(u) = e^{3u} = \frac{x^3+y^3}{x^2+y^2}$ is homogeneous with deg 1.

(i) By cor - 2

(i) By cor - 2

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 1\cdot\frac{e^{3u}}{3e^{3u}} = \frac{1}{3}$$

(ii) By cor - 3

$$g(u) = n\frac{f(u)}{f'(u)} = \frac{1}{3} \qquad \therefore g'(u) = 0.$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

$$= \frac{1}{3}[0-1]$$

$$= -\frac{1}{3}$$