# Corollary -1

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**Corollary 1** If z is a homogeneous function of two variables x and y of degree n then

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$

* If u is a homogeneous function of 3 variables x, y, z of degree n then

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2yz\frac{\partial^2 u}{\partial y\partial z}+2xz\frac{\partial^2 u}{\partial x\partial z}=n(n-1)u$$

**SOME SOLVED EXAMPLES:**

**1.** If $u=(x/y)^{y/x}$, prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$

Soln :- $f(xt,yt)=\left(\frac{xt}{yt}\right)^{yt/xt}=\left(\frac{x}{y}\right)^{\frac{y}{x}}=t^0 f(x,y)$

$\therefore$ u is homogeneous function of deg $n=0$

$\therefore$ By corollary - 1

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u=0.$$

**2.** If $u=\frac{x^3y+y^3x}{y-x}$, prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=6\left(\frac{x^3y+y^3x}{y-x}\right)$

Soln u is homogeneous function of deg $n=3$

$\therefore$ By corollary- 1.

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u=3(3-1)u=6u.$$

**3.** If $u=x^2tan^{-1}\frac{y}{x}+y^2sin^{-1}\frac{x}{y}$, prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2u$

Soln :- u is homogeneous with deg $n=2$

$\therefore$ By corollary-1

$$LHS = n(n-1)u = 2(2-1)u = 2u$$

**4.** If $z = x^n f\left(\frac{y}{x}\right) + y^{-n} f\left(\frac{x}{y}\right)$, prove that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$

Soln:- $z$ is not homogeneous

but $z = u + v$

$u = x^n f\left(\frac{y}{x}\right)$ is homogeneous with deg $n$

$v = y^{-n} f\left(\frac{x}{y}\right)$ is homogeneous with deg$(-n)$

By Euler's thm,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{——} \ (1)$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -nv \quad \text{——} \ (2)$$

By corollary-1,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = (n^2-n)u \quad \text{——} \ (3)$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = (-n)(-n-1)v = (n^2+n)v \quad \text{——} \ (4)$$

adding (1), (2), (3), (4)

$$x^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x\partial y}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$+ x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = nu - nv + (n^2-n)u + (n^2+n)v$$

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2u + n^2v$$

$$= n^2(u+v)$$

$$= n^2 z$$

HW.

**5.** If $u = x^3 sin^{-1}\frac{y}{x} + x^4 tan^{-1}\frac{y}{x}$, find the value of $\underline{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}} + \underline{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}}$ $at\ x = 1, y = 1$

Soln:- $u$ is not homogeneous

but $u = v + w$

$v = x^3 \sin^{-1}\left(\frac{y}{x}\right)$ is homogeneous deg $n_1 = 3$

$w = x^4 \tan^{-1}\left(\frac{y}{x}\right)$ is homogeneous deg $n_2 = 4$

Apply Euler's thm to $v$ & $w$

— (1)

— (2)

Apply cor-1 to $v$ & $w$

— (3)

— (4)

Add (1), (2), (3), (4)

**6.** If $u = y^2 e^{\frac{y}{x}} + x^2 \tan^{-1}\left(\frac{x}{y}\right)$, show that $\underline{xu_x + yu_y} = 2u$ and $\underline{x^2u_{xx}} + 2xy\underline{u_{xy}} + y^2\underline{u_{yy}} = 2u$

Soln:- u is homogeneous with deg $n=2$ (show this)

$u_x \to \frac{\partial u}{\partial x}$, $u_y \to \frac{\partial u}{\partial y}$, $u_{xx} \to \frac{\partial^2 u}{\partial x^2}$, $u_{xy} \to \frac{\partial^2 u}{\partial x \partial y}$, $u_{yy} \to \frac{\partial^2 u}{\partial y^2}$

$\therefore$ By Euler's thm

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \implies xu_x + yu_y = 2u$$

By corollary - 1

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = 2(2-1)u = 2u.$$