HOMOGENEOUS FUNCTIONS

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Homogeneous Functions:

A function $u = f(x, y)$ is said to be homogeneous function of degree n, if $u = x^n f\left(\frac{y}{x}\right)$ where, n is real number

Note: Degree of a homogeneous function $u = f(x, y)$ can be obtained by replacing x by xt and y by yt and if $f(xt, yt) = t^n f(x, y) = t^n u$ then u is a homogeneous function of degree n.

Same method can be extended for a function of more than two variables

A function $f(x, y, z)$ is called a homogeneous function of degree n if by putting $X = xt, Y = yt, Z = zt$ the function $f(X, Y, Z)$ becomes $t^n f(x, y, z)$ i.e. $f(xt, yt, zt) = t^n f(x, y, z)$

e.g. $f(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ is a homogeneous function of degree 2.

EULER'S THEOREM:

If u is a homogeneous function of two variables x and y of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

EULER'S THEOREM FOR A FUNCTION OF THREE VARIABLES:

Theorem: If u is a homogeneous function of three variables x, y, z of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

Note: In general, if u is a homogeneous function of x, y, z ….. t of degree n then Euler's Theorem states that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \cdots + t\frac{\partial u}{\partial t} = nu$$

SOME SOLVED EXAMPLES:

1. If $u = e^{x/y}\sin\left(\frac{x}{y}\right) + e^{y/x}\cos\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

$$u = f(x, y) = e^{x/y}\sin\left(\frac{x}{y}\right) + e^{y/x}\cos\left(\frac{y}{x}\right)$$

put $x \to xt$, $y \to yt$

$$f(xt, yt) = e^{xt/yt}\sin\left(\frac{xt}{yt}\right) + e^{yt/xt}\cos\left(\frac{yt}{xt}\right)$$

$$= e^{x/y}\sin\left(\frac{x}{y}\right) + e^{y/x}\cos\left(\frac{y}{x}\right)$$

$$= t^0 f(x, y)$$

$\therefore$ $u = f(x, y)$ is a homogeneous function of deg $n = 0$.

$\therefore$ By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 0.$$

2. If $u = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u = 0$.

Soln :- $u = f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}$

put $x = xt$, $y = yt$

$$f(xt, yt) = \frac{1}{(xt)^2} + \frac{1}{(xt)(yt)} + \frac{\log\left(\frac{xt}{yt}\right)}{(xt)^2 + (yt)^2}$$

$$= \frac{1}{x^2t^2} + \frac{1}{xyt^2} + \frac{\log\left(\frac{x}{y}\right)}{t^2(x^2+y^2)}$$

$$= \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}\right]$$

$$f(xt, yt) = t^{-2} f(x,y)$$

$\therefore$ $u = f(x,y)$ is homogeneous with deg $n = -2$

$\therefore$ By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = -2u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u = 0.$$ Hence proved.

3. If $u = \frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Soln :- $u = f(x,y) = \frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$

$$f(xt, yt) = \frac{\sqrt{xt \cdot yt}}{\sqrt{xt} + \sqrt{yt}} = \frac{t\sqrt{xy}}{\sqrt{t}\left[\sqrt{x}+\sqrt{y}\right]} = t^{1/2}\frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$$

$$\therefore f(xt, yt) = t^{1/2} f(x,y)$$

$\therefore f(xt, yt) = t^{1/2} f(x,y)$

$\therefore u$ is homogeneous function with deg $n = \frac{1}{2}$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = \frac{1}{2}u = \frac{1}{2}\frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$$

H.W

4. If $u = (8x^2 + y^2)(\log x - \log y)$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

5. (i) Verify Euler's Theorem for $u = ax^2 + 2hxy + by^2$

Soln :- u is homogeneous function of deg 2

(check this)

To verify Euler's theorem, we have to show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

$u = ax^2 + 2hxy + by^2$

$\frac{\partial u}{\partial x} = 2ax + 2hy \quad \Rightarrow x\frac{\partial u}{\partial x} = 2ax^2 + 2hxy$

$\frac{\partial u}{\partial y} = 2hx + 2by \quad \Rightarrow y\frac{\partial u}{\partial y} = 2hxy + 2by^2$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2ax^2 + 2hxy + 2hxy + 2by^2$$

$$= 2(ax^2 + 2hxy + by^2) = 2u$$

$\therefore$ Euler's thm is verified.

H.W

(ii) Verify Euler's Theorem for $u = \frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$

we can show that u is homogeneous function with deg $n = \frac{1}{2}$

Hence to verify Euler's theorem, we show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{2}u$$

Hence to verify ...

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{1}{2}u$$

Now $u=\dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$

$$\therefore \frac{\partial u}{\partial x}=\frac{(\sqrt{x}+\sqrt{y}+\sqrt{z})\cdot(1)-(x+y+z)\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2}$$

$$=\frac{2\sqrt{x}(\sqrt{x}+\sqrt{y}+\sqrt{z})-(x+y+z)}{2\sqrt{x}(\sqrt{x}+\sqrt{y}+\sqrt{z})^2}$$

$$\therefore x\frac{\partial u}{\partial x}=\frac{2x(\sqrt{x}+\sqrt{y}+\sqrt{z})-\sqrt{x}(x+y+z)}{2(\sqrt{x}+\sqrt{y}+\sqrt{z})^2} \quad \text{——} (1)$$

Similarly,

$$y\frac{\partial u}{\partial y}=\frac{2y(\sqrt{x}+\sqrt{y}+\sqrt{z})-\sqrt{y}(x+y+z)}{2(\sqrt{x}+\sqrt{y}+\sqrt{z})^2} \quad \text{——} (2)$$

$$z\frac{\partial u}{\partial z}=\frac{2z(\sqrt{x}+\sqrt{y}+\sqrt{z})-\sqrt{z}(x+y+z)}{2(\sqrt{x}+\sqrt{y}+\sqrt{z})^2} \quad \text{——} (3)$$

adding (1), (2) 4 (3)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{2(x+y+z)(\sqrt{x}+\sqrt{y}+\sqrt{z})-(x+y+z)(\sqrt{x}+\sqrt{y}+\sqrt{z})}{2(\sqrt{x}+\sqrt{y}+\sqrt{z})^2}$$

$$=\frac{x+y+z}{2(\sqrt{x}+\sqrt{y}+\sqrt{z})}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{1}{2}u$$

Hence Euler's theorem is verified.

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(iii) Verify Euler's Theorem for $u = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$

Soln:- $u = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$ is homogeneous function with deg $n = 6$.

To verify Euler's theorem, we show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6u$$

Now $u = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$

$$\therefore \frac{\partial u}{\partial x} = y^2\left[4x^3 \sin^{-1}\left(\frac{y}{x}\right) + x^4 \cdot \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \left(\frac{-y}{x^2}\right)\right]$$

$$\frac{\partial u}{\partial x} = y^2\left[4x^3 \sin^{-1}\left(\frac{y}{x}\right) - \frac{yx^3}{\sqrt{x^2-y^2}}\right]$$

$$\therefore x\frac{\partial u}{\partial x} = 4x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right) - \frac{y^3 x^4}{\sqrt{x^2-y^2}} \quad \text{——} \ (1)$$

$u = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$

$$\text{Now, } \frac{\partial u}{\partial y} = x^4\left[2y \sin^{-1}\left(\frac{y}{x}\right) + y^2 \cdot \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \left(\frac{1}{x}\right)\right]$$

$$= x^4\left[2y \sin^{-1}\left(\frac{y}{x}\right) + \frac{y^2}{\sqrt{x^2-y^2}}\right]$$

$$\therefore y\frac{\partial u}{\partial y} = 2y^2 x^4 \sin^{-1}\left(\frac{y}{x}\right) + \frac{y^3 x^4}{\sqrt{x^2-y^2}} \quad \text{——} \ (2)$$

adding (1) & (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right) = 6u$$

Hence Euler's theorem is verified.

$x^3y^3z^3$ $(xy+yz+zx)$ ∂u ∂u ∂u

6. If $u = \frac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Soln:- u is not homogeneous

but $u = v + w$

where $v = \frac{x^3y^3z^3}{x^3+y^3+z^3}$ & $w = \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$

v is homogeneous with deg $n_1 = 6$

w is homogeneous with deg $n_2 = 0$

By Euler's thm,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = n_1 v = 6v \quad \text{——— (1)}$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = n_2 w = 0\cdot w \quad \text{——— (2)}$$

$u = v + w$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}$$

adding (1) & (2)

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) + z\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\right) = 6v + 0w$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6v = 6\left(\frac{x^3y^3z^3}{x^3+y^3+z^3}\right)$$

7. If u is a function of a homogeneous function i.e. if $u = f(v)$ where v is a homogeneous function x, y of degree n, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,vf'(v)$ hence, deduce that, if $u = \log v$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n$.

Soln:- $u = f(v)$ where v is homogeneous of deg n.

By Euler's thm

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv \quad \text{——— (1)}$$

By Euler's thm

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv \quad \text{———} \quad (1)$$

Now $u = f(v)$

$$\therefore \frac{\partial u}{\partial x} = f'(v)\cdot\frac{\partial v}{\partial x} \Rightarrow x\frac{\partial u}{\partial x} = xf'(v)\frac{\partial v}{\partial x} \quad -(2)$$

$$\frac{\partial u}{\partial y} = f'(v)\frac{\partial v}{\partial y} \Rightarrow y\frac{\partial u}{\partial y} = yf'(v)\frac{\partial v}{\partial y} \quad -(3)$$

adding (2) & (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = f'(v)\left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right]$$

$$= f'(v)[nv]$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nvf'(v)$$

If $u = \log v = f(v)$

$$\therefore f'(v) = \frac{1}{v}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nvf'(v) = nv\cdot\frac{1}{v} = n$$

Hence prove.