Similarly, we have,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy},\qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y\,\partial x}=f_{yx},$$

and $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\,\partial y}=f_{xy}.$

It may be noted that although $\frac{\partial^2 z}{\partial x\,\partial y}$ and $\frac{\partial^2 z}{\partial y\,\partial x}$ are equal in general, they need not be equal always.

5. Partial Derivatives of Some Standard Functions

Using the above definition *i.e.* treating y constant while partially differentiating z w.r.t. x and treating x constant while partially differentiating z w.r.t. y, we can write down partial derivatives of some standard functions.

1. If $z=k$, $\frac{\partial z}{\partial x}=0$, $\frac{\partial z}{\partial y}=0$.

   If $z=f(y)$, $\frac{\partial z}{\partial x}=0$ because $f(y)$ is constant for partial differentiation w.r.t. x.

   If $z=f(x)$, $\frac{\partial z}{\partial y}=0$ because $f(x)$ is constant for partial differentiation w.r.t. y.

2. If $z=x^n y^m$, $\frac{\partial z}{\partial x}=nx^{n-1}\cdot y^m$; $\frac{\partial z}{\partial y}=x^n\cdot my^{m-1}$

   For example, if $z=x^2y^3$, $\frac{\partial z}{\partial x}=2xy^3$, $\frac{\partial z}{\partial y}=3x^2y^2$.

3. If $z=\sin(x+y)$, $\frac{\partial z}{\partial x}=\cos(x+y)$; $\frac{\partial z}{\partial y}=\cos(x+y)$

4. If $z=e^{x+y}$, $\frac{\partial z}{\partial x}=e^{x+y}$; $\frac{\partial z}{\partial y}=e^{x+y}$

5. If $z=\log(x+y)$, $\frac{\partial z}{\partial x}=\frac{1}{x+y}$; $\frac{\partial z}{\partial y}=\frac{1}{x+y}$

6. If $z=\sin^{-1}\left(\frac{x}{y}\right)$,

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-(x^2/y^2)}}\cdot\left(\frac{1}{y}\right)=\frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-(x^2/y^2)}}\cdot\left(-\frac{x}{y^2}\right)=-\frac{x}{y\sqrt{y^2-x^2}}$$

7. If $z=\tan^{-1}\left(\frac{x}{y}\right)$,

$$\frac{\partial z}{\partial x}=\frac{1}{1+(x^2/y^2)}\cdot\frac{1}{y}=\frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y}=\frac{1}{1+(x^2/y^2)}\cdot\left(-\frac{x}{y^2}\right)=-\frac{x}{y^2+x^2}$$

8. If $z=x^y$, $\frac{\partial z}{\partial x}=yx^{y-1}$; $\frac{\partial z}{\partial y}=x^y\cdot\log x$. **[Note this]**

Standard Rules

If u and v are functions of x and y possessing partial derivatives of the first order, then we can use standard rules of differentiation of sum, difference, product and quotient of u and v as stated below.

1. If $z = u \pm v$, $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial u}{\partial x} \pm \dfrac{\partial v}{\partial x}, \quad \dfrac{\partial z}{\partial y} = \dfrac{\partial u}{\partial y} \pm \dfrac{\partial v}{\partial y}$

2. If $z = uv$, $\quad \dfrac{\partial z}{\partial x} = u\dfrac{\partial v}{\partial x} + v\dfrac{\partial u}{\partial x}, \quad \dfrac{\partial z}{\partial y} = u\dfrac{\partial v}{\partial y} + v\dfrac{\partial u}{\partial y}$

3. If $z = \dfrac{u}{v}$, $\quad \dfrac{\partial z}{\partial x} = \dfrac{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}}{v^2}, \quad \dfrac{\partial z}{\partial y} = \dfrac{v\dfrac{\partial u}{\partial y} - u\dfrac{\partial v}{\partial y}}{v^2}$

Type I : Partial Differentiation using Standard Rules

Example 1 : If $z = ax^2 + by^2 + 2ab\,xy$, find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

Sol. : We have $\dfrac{\partial z}{\partial x} = 2ax + 2aby \quad ; \quad \dfrac{\partial z}{\partial y} = 2by + 2abx.$

Example 2 : If $u = e^x \sin x \sin y$, find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$.

Sol. :

$$\frac{\partial u}{\partial x} = e^x \sin x \sin y + e^x \cos x \sin y$$

$$\frac{\partial u}{\partial y} = e^x \sin x \cos y$$

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : If $z(x + y) = x - y$, find $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2$. **(M.U. 2016)**

Sol. : We have $z = \dfrac{x - y}{x + y}$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{(x + y)(1) - (x - y)(1)}{(x + y)^2} = \frac{2y}{(x + y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x + y)(-1) - (x - y)(1)}{(x + y)^2} = -\frac{2x}{(x + y)^2}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{2y + 2x}{(x + y)^2}\right]^2 = \frac{4}{(x + y)^2}$$

Example 2 (a) : If $z(x + y) = (x^2 + y^2)$, prove that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$ **(M.U. 1991, 98, 2002)**

Sol. : Since $z = \dfrac{(x^2 + y^2)}{x + y}$,

$$\frac{\partial z}{\partial x}=\frac{(x+y)2x-(x^2+y^2)}{(x+y)^2}=\frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y}=\frac{(x+y)2y-(x^2+y^2)}{(x+y)^2}=\frac{-x^2+2xy+y^2}{(x+y)^2}$$

$$\therefore \quad \text{l.h.s.}=\left[\frac{x^2+2xy-y^2+x^2-2xy-y^2}{(x+y)^2}\right]^2=\left[2\cdot\frac{(x^2-y^2)}{(x+y)^2}\right]^2$$

$$=\left[2\cdot\left(\frac{x-y}{x+y}\right)\right]^2=4\frac{(x-y)^2}{(x+y)^2}$$

$$\therefore \quad \text{r.h.s.}=4\left[1-\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{-x^2+2xy+y^2}{(x+y)^2}\right]$$

$$=4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right]=4\frac{(x-y)^2}{(x+y)^2}$$

$\therefore$ l.h.s. = r.h.s.

Example 3 (a) : If $u=\tan^{-1}\frac{y}{x}$, find the value of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$. **(M.U. 2010, 14)**

Sol. : We have

$$\frac{\partial u}{\partial x}=\frac{1}{1+(y^2/x^2)}\cdot\left(\frac{-y}{x^2}\right)=-\frac{y}{x^2+y^2};\quad \frac{\partial u}{\partial y}=\frac{1}{1+(y^2/x^2)}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}$$

Now, $$\frac{\partial^2 u}{\partial x^2}=-y\cdot\frac{-1}{(x^2+y^2)^2}(2x);\qquad \frac{\partial^2 u}{\partial y^2}=x\cdot\frac{-1}{(x^2+y^2)}\cdot(2y)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Example 4 (a) : If $x=\cos\theta-r\sin\theta$, $y=\sin\theta+r\cos\theta$, prove that $\frac{\partial r}{\partial x}=\frac{x}{r}$. **(M.U. 2014)**

Sol. : Squaring, we get

$$x^2+y^2=\cos^2\theta+r^2\sin^2\theta-2r\sin\theta\cos\theta+\sin^2\theta+r^2\cos^2\theta+2r\sin\theta\cos\theta$$

$$=(\cos^2\theta+\sin^2\theta)+r^2(\cos^2\theta+\sin^2\theta)=1+r^2$$

$$\therefore \quad r^2=x^2+y^2-1$$

Differentiating this partially w.r.t. x,

$$2r\frac{\partial r}{\partial x}=2x \qquad \therefore \quad \frac{\partial r}{\partial x}=\frac{x}{r}.$$

EXERCISE - II

Class (a) : 3 Marks

1. Find the partial derivatives of the following functions.

1. $x^2y^3+x^3y^2$ **2.** $2x^2+3xy+y^2$ **3.** $\log x\cdot\sin y$ **4.** $\sin x\cos y$

5. $\dfrac{\sin x}{\cos y}$ 6. $e^x \sin y$ 7. $10^x \cdot \cos y$ 8. $3^x \cdot \tan y$

9. $\dfrac{x}{x^2+y^2}$ 10. $\dfrac{y}{x^2+y^2}$ 11. $2^x \sin y \cos z$ 12. $e^x y^3 z^2$.

[**Ans. :** (1) $\dfrac{\partial u}{\partial x} = 2xy^3 + 3x^2y^2$, $\dfrac{\partial u}{\partial y} = 3x^2y^2 + 2x^3y$; (2) $\dfrac{\partial u}{\partial x} = 4x + 3y$, $\dfrac{\partial u}{\partial y} = 3x + 2y$;

(3) $\dfrac{\partial u}{\partial x} = \dfrac{\sin y}{x}$, $\dfrac{\partial u}{\partial y} = \log x \cos y$; (4) $\dfrac{\partial u}{\partial x} = \cos x \cos y$, $\dfrac{\partial u}{\partial y} = -\sin x \sin y$;

(5) $\dfrac{\partial u}{\partial x} = \dfrac{\cos x}{\cos y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\sin x}{\cos^2 y} \cdot \sin y$; (6) $\dfrac{\partial u}{\partial x} = e^x \sin y$, $\dfrac{\partial u}{\partial y} = e^x \cos y$;

(7) $\dfrac{\partial u}{\partial x} = 10^x \log 10 \cdot \cos y$, $\dfrac{\partial u}{\partial y} = -10^x \sin y$; (8) $\dfrac{\partial y}{\partial x} = 3^x \log 3 \tan y$, $\dfrac{\partial u}{\partial y} = 3^x \sec^2 y$;

(9) $\dfrac{\partial u}{\partial x} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$, $\dfrac{\partial u}{\partial y} = -\dfrac{2xy}{(x^2+y^2)^2}$; (10) $\dfrac{\partial u}{\partial x} = -\dfrac{-2xy}{(x^2+y^2)^2}$, $\dfrac{\partial u}{\partial y} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$;

(11) $\dfrac{\partial u}{\partial x} = 2^x \log 2 \cdot \sin y \cos z + 2^x \cos y \cos z - 2^x \sin y \sin z$;

(12) $\dfrac{\partial u}{\partial x} = e^x y^3 z^2 + 3e^x y^2 z^2 + 2e^x y^3 z$.]

2. Find the second order partial derivatives $\dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial y^2}$ of the following functions.

1. $x^3 y + xy^3$ 2. $e^x \cdot y^2$
3. $x^2 - 4x^2 y + 5y^2$ 4. $e^x \log y + \sin y \log x$.

[**Ans. :** (1) $\dfrac{\partial^2 u}{\partial x^2} = 6xy$, $\dfrac{\partial^2 u}{\partial y^2} = 6xy$; (2) $\dfrac{\partial^2 u}{\partial x^2} = e^x \cdot y^2$, $\dfrac{\partial^2 u}{\partial y^2} = 2e^x$;

(3) $\dfrac{\partial^2 u}{\partial x^2} = 2 - 8y$, $\dfrac{\partial^2 u}{\partial y^2} = 10$;

(4) $\dfrac{\partial^2 u}{\partial x^2} = e^x \log y - \dfrac{\sin y}{x^2}$, $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{e^x}{y^2} - \sin y \cdot \log x$.]

Class (a) : 3 Marks

1. If $u = e^{ax} \sin by$, prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

2. If $u = \sin^{-1} \dfrac{x}{y}$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 0$.

3. If $u = \dfrac{x}{y} + \dfrac{y}{x}$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 0$.

4. If $u = x^2 y + y^2 z + z^2 x$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} = 3u$.

(Examples 2, 3 and 4 can also be solved by using Eulers theorem. See Chapter 7.)

5. If $u = \tan^{-1} \dfrac{y}{x}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

6. Differentiation of a Function of a Function

Let $z = f(u)$ and $u = \Phi(x, y)$ so that z is function of u, and u itself is a function of two independent variables x and y. The two relations define z as a function of x and y. In such cases z may be called **a function of a function** of x and y.

e.g. (i) $z = \dfrac{1}{u}$ and $u = \sqrt{x^2 + y^2}$. (ii) $z = \tan u$ and $u = x^2 + y^2$

define z as a function of a function of x and y.

Differentiation : If $z = f(u)$ is differentiable function of u and $u = \Phi(x, y)$ possesses first order partial derivatives, then

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \qquad \text{i.e.} \qquad \frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = f'(u) \frac{\partial u}{\partial y}.$$

e.g. If $z = (ax + by)^n$, then

$$\frac{\partial z}{\partial x} = n(ax + by)^{n-1} \cdot a \quad \text{and} \quad \frac{\partial z}{\partial y} = n(ax + by)^{n-1} \cdot b$$

The rule can be easily remembered with the help of the tree diagram given on the right.

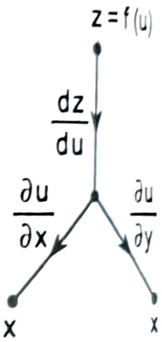

Fig. 6.2

We consider below some standard functions of the type $z = f(u)$.

1. If $z = u^n$, then $\dfrac{\partial z}{\partial x} = nu^{n-1} \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y} = nu^{n-1} \dfrac{\partial u}{\partial y}$.

   e.g., if $z = (2x + 3y)^5$, then

   $$\frac{\partial z}{\partial x} = 5(2x + 3y)^4 \cdot 2 \quad \text{and} \quad \frac{\partial z}{\partial y} = 5(2x + 3y)^4 \cdot 3$$

2. If $z = \sqrt{u}$, then $\dfrac{\partial z}{\partial x} = \dfrac{1}{2\sqrt{u}} \cdot \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y} = \dfrac{1}{2\sqrt{u}} \cdot \dfrac{\partial u}{\partial y}$

   e.g., if $z = \sqrt{(4x - 5y)}$, then

   $$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{(4x - 5y)}} \cdot 4 \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{(4x - 5y)}} \cdot (-5)$$

3. If $z = \sin u$, then $\dfrac{\partial z}{\partial x} = \cos u \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y} = \cos u \dfrac{\partial u}{\partial y}$.

   e.g., if $z = \sin(2x - y)$, then

   $$\frac{\partial z}{\partial x} = \cos(2x - y) \cdot 2 \quad \text{and} \quad \frac{\partial z}{\partial y} = \cos(2x - y)(-1)$$

4. If $z = \cos u$, then $\dfrac{\partial z}{\partial x} = -\sin u \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y} = -\sin u \dfrac{\partial u}{\partial y}$.

   e.g. if $z = \cos(3x - 2y)$, then

   $$\frac{\partial z}{\partial x} = -\sin(3x - 2y) \cdot (3) \quad \text{and} \quad \frac{\partial z}{\partial y} = -\sin(3x - 2y)(-2)$$

5. If $z = \tan u$, then $\dfrac{\partial z}{\partial x} = \sec^2 u \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y} = \sec^2 u \dfrac{\partial u}{\partial y}$.

e.g., If $z = \tan(3x + 2y)$, then

$$\frac{\partial z}{\partial x} = \sec^2(3x+2y) \cdot 3 \quad \text{and} \quad \frac{\partial z}{\partial y} = \sec^2(3x+2y) \cdot 2$$

6. If $z = e^u$, then $\frac{\partial z}{\partial x} = e^u \frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = e^u \frac{\partial u}{\partial y}$.

e.g., if $z = e^{3x-4y}$, then

$$\frac{\partial z}{\partial x} = e^{3x-4y} \cdot 3 \quad \text{and} \quad \frac{\partial z}{\partial y} = e^{3x-4y}(-4)$$

7. If $z = \log u$, then $\frac{\partial z}{\partial x} = \frac{1}{u} \cdot \frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{1}{u} \cdot \frac{\partial u}{\partial y}$.

e.g., if $z = \log(3x + 7y)$, then

$$\frac{\partial z}{\partial x} = \frac{1}{(3x+7y)} \cdot 3 \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{1}{(3x+7y)} \cdot 7$$

Type II : Partial Derivatives of First Order of a Function of a Function : Class (a) : 3 Marks

Example 1 (a) : If $u = \cos(\sqrt{x} + \sqrt{y})$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\sin\left(\sqrt{x}+\sqrt{y}\right) = 0.$$

Sol. : We have $\frac{\partial u}{\partial x} = -\sin\left(\sqrt{x}+\sqrt{y}\right)\frac{1}{2\sqrt{x}}$; $\frac{\partial u}{\partial y} = -\sin\left(\sqrt{x}+\sqrt{y}\right)\frac{1}{2\sqrt{y}}$.

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\sin\left(\sqrt{x}+\sqrt{y}\right) \cdot \frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\sin\left(\sqrt{x}+\sqrt{y}\right) = 0.$$

Example 2 (a) : If $u = \sin\left(\sqrt{x}+\sqrt{y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\cos\left(\sqrt{x}+\sqrt{y}\right)$.

Sol. : Prove it.

(For another method to solve this example, see Ex. 10, page 7-11.)

Example 3 (a) : If $z = e^{ax+by} f(ax - by)$, prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

Sol. : Let $ax + by = u$ and $ax - by = v$

$$\therefore \quad \frac{\partial u}{\partial x} = a, \quad \frac{\partial u}{\partial y} = b, \quad \frac{\partial v}{\partial x} = a, \quad \frac{\partial v}{\partial y} = -b$$

Hence, $z = e^u \cdot f(v)$,

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[e^u\right] f(v) + e^u \cdot \frac{\partial}{\partial x} f(v) = e^u \frac{\partial u}{\partial x} \cdot f(v) + e^u \cdot f'(v)\frac{\partial v}{\partial x}$$

$$= e^u \cdot a \cdot f(v) + e^u \cdot f'(v) \cdot a$$

Also, $\frac{\partial z}{\partial y} = e^u \frac{\partial u}{\partial y} \cdot f(v) + e^u \cdot f'(v)\frac{\partial v}{\partial y} = e^u \cdot b \cdot f(v) + e^u \cdot f'(v) \cdot (-b)$

$$\therefore \quad b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abe^u f(v) = 2abz.$$

Example 4 (a) : If $u = (1 - 2xy + y^2)^{-1/2}$, prove that

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2 u^3.$$

(M.U. 1991, 99, 2004, 05, 08)

Sol. : Since $u = (1 - 2xy + y^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2y) = yu^3 \qquad \therefore \quad x\frac{\partial u}{\partial x} = xy\,u^3$$

Also, $$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2x + 2y) = (x - y)\,u^3 \qquad \therefore \quad y\frac{\partial u}{\partial y} = (xy - y^2)\,u^3$$

$$\therefore \quad x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xy\,u^3 - xy\,u^3 + y^2\,u^3 = y^2u^3.$$

Example 5 (a) : If $u = \log(\tan x + \tan y)$, prove that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} = 2.$$

(M.U. 1991, 2003, 05, 10, 12, 15)

Sol. : We have $$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y}\sec^2 x$$

$$\therefore \quad \sin 2x\frac{\partial u}{\partial x} = 2\sin x\cos x\frac{1}{(\tan x + \tan y)}\cdot\sec^2 x = 2\cdot\frac{\tan x}{\tan x + \tan y}$$

Similarly, $$\sin 2y\frac{\partial u}{\partial y} = 2\cdot\frac{\tan y}{\tan x + \tan y}.$$

$$\therefore \quad \sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} = 2\cdot\frac{\tan x + \tan y}{\tan x + \tan y} = 2.$$

Similarly, prove that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2.$$

Example 6 (a) : If $u = f[x^2 + y^2 + z^2]$, $x = r\cos\alpha\cos\beta$, $y = r\cos\alpha\sin\beta$, $z = r\sin\alpha$, show that

$$\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial \beta} = 0.$$

(M.U. 1988)

Sol. : From data,

$$x^2 + y^2 + z^2 = r^2\cos^2\alpha\cos^2\beta + r^2\cos^2\alpha\sin^2\beta + r^2\sin^2\alpha$$

$$\therefore \quad x^2 + y^2 + z^2 = r^2\cos^2\alpha + r^2\sin^2\alpha = r^2$$

$$\therefore \quad u = f[x^2 + y^2 + z^2] = f[r^2]$$

$$\therefore \quad \frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial \beta} = 0$$

($\because$ u is independent of α and β)

Example 7 (a) : If $u = \dfrac{e^{x+y}}{e^x + e^y}$, prove that $u_x + u_y = u$.

(M.U. 1996, 2000)

Sol. : We have $$\frac{\partial u}{\partial x} = \frac{(e^x + e^y)\cdot e^{x+y} - e^{x+y}\cdot e^x}{(e^x + e^y)^2} = \frac{e^{x+y}(e^y)}{(e^x + e^y)^2}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{e^{x+y}(e^x)}{(e^x + e^y)^2}$$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\frac{e^{x+y}\cdot(e^x+e^y)}{(e^x+e^y)^2}=\frac{e^{x+y}}{e^x+e^y}=u.$$

Similarly, prove that if $u=\dfrac{e^{x+y+z}}{e^x+e^y+e^z}$, then $u_x+u_y+u_z=2u$.

Class (b) : 6 Marks

Example 1 (b) : If $\theta=t^n e^{-r^2/4t}$, find n which will make

$$\frac{\partial\theta}{\partial t}=\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right).$$ **(M.U. 1986, 93, 2000, 02, 06)**

Sol. :

$$\frac{\partial\theta}{\partial t}=n\,t^{n-1}\cdot e^{-r^2/4t}+t^n e^{-r^2/4t}\cdot\left(-\frac{r^2}{4}\right)\left(-\frac{1}{t^2}\right)$$

$$=\frac{n}{t}\cdot t^n\cdot\frac{\theta}{t^n}+t^n\cdot\frac{\theta}{t^n}\left(\frac{r^n}{4t^2}\right)$$

$$=\frac{n}{t}\theta+\frac{r^2}{4t^2}\theta=\left(\frac{n}{t}+\frac{r^2}{4t^2}\right)\theta \qquad \left[\because\ e^{-r^2/4t}=\frac{\theta}{t^n}\right] \qquad \text{.................... (1)}$$

Also, $\dfrac{\partial\theta}{\partial r}=t^n e^{-r^2/4t}\cdot\left(-\dfrac{2r}{4t}\right)=-\dfrac{r\theta}{2t} \qquad \therefore\ r^2\dfrac{\partial\theta}{\partial r}=-\dfrac{r^3\theta}{2t}$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial}{\partial r}\left(-\frac{r^3\theta}{2t}\right)=-\frac{1}{2t}\cdot\frac{\partial}{\partial r}(r^3\theta)=-\frac{1}{2t}\left[r^3\frac{\partial\theta}{\partial r}+3r^2\theta\right]$$

$$=-\frac{1}{2t}\left[-\frac{r^4\theta}{2t}+3r^2\theta\right]=r^2\left(\frac{r^2}{4t^2}-\frac{3}{2t}\right)\theta$$

$$\therefore \quad \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\left(\frac{r^2}{4t^2}-\frac{3}{2t}\right)\theta \qquad \text{.................... (2)}$$

$\therefore$ Equating (1) and (2), we get

$$\frac{n}{t}=-\frac{3}{2t} \qquad \therefore\ n=-\frac{3}{2}.$$

Example 2 (b) : Find the value of n so that $V=r^n(3\cos^2\theta-1)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0.$$ **(M.U. 1995, 2001, 02, 06)**

Sol. : We have by differentiating partially w.r.t. r,

$$\frac{\partial V}{\partial r}=n\,r^{n-1}(3\cos^2\theta-1) \qquad \therefore\ r^2\frac{\partial V}{\partial r}=n\,r^{n+1}(3\cos^2\theta-1)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=n(n+1)\,r^n(3\cos^2\theta-1) \qquad \text{.................... (1)}$$

Further differentiating partially w.r.t. θ,

$$\frac{\partial V}{\partial\theta}=r^n(-6\cos\theta\sin\theta) \qquad \therefore\ \sin\theta\frac{\partial V}{\partial\theta}=-6\,r^n\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = -6\,r^n\,[\,2\sin\theta\cos^2\theta - \sin^3\theta\,]$$

$$\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = -6\,r^n\,(2\cos^2\theta - \sin^2\theta)$$

$$= -6\,r^n\,(3\cos^2\theta - 1) \qquad \text{.................. (2)}$$

Adding (1) and (2) and equating the result to zero, (by data) we get,

$$\therefore \quad n\,(n+1)\,r^n\,(3\cos^2\theta - 1) - 6\,r^n\,(3\cos^2\theta - 1) = 0$$

$$\therefore \quad [\,n\,(n+1) - 6\,]\,r^n\,(3\cos^2\theta - 1) = 0$$

$$\therefore \quad n^2 + n - 6 = 0 \quad \therefore \quad (n+3)\,(n-2) = 0 \quad \therefore \quad n = 2 \text{ or } -3.$$

Type III : Partial Derivatives of Second Order of a Function of a Function

Class (a) : 3 Marks

Example 1 (a) : If $u = \log(x^2 + y^2)$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$. (M.U. 2013)

Sol. : We have $\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}\cdot 2x$ and $\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2}\cdot 2y$

$$\therefore \quad \frac{\partial^2 u}{\partial x\,\partial y} = 2x\left[-\frac{1}{(x^2 + y^2)^2}\right]\cdot 2y = -\frac{4xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 2y\left[-\frac{1}{(x^2 + y^2)^2}\right]\cdot 2x = -\frac{4xy}{(x^2 + y^2)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}.$$

Example 2 (a) : If $u = 2\,(ax + by)^2 - k\,(x^2 + y^2)$ and $a^2 + b^2 = k$, evaluate $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

Sol. : We have $\frac{\partial u}{\partial x} = 4\,(ax + by)\,a - 2\,kx \quad \therefore \quad \frac{\partial^2 u}{\partial x^2} = 4a^2 - 2k$

and $\frac{\partial u}{\partial y} = 4\,(ax + by)\,b - 2\,ky \quad \therefore \quad \frac{\partial^2 u}{\partial y^2} = 4b^2 - 2k$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\,(a^2 + b^2) - 4k = 4k - 4k = 0 \quad [\because a^2 + b^2 = k]$$

Example 3 (a) : If $z = \tan(y + ax) + (y - ax)^{3/2}$, show that $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$.

(M.U. 2002, 03, 09, 11, 17)

Sol. : We have $\frac{\partial z}{\partial x} = a\cdot\sec^2(y + ax) - a\cdot\frac{3}{2}(y - ax)^{1/2}$

and $\frac{\partial^2 z}{\partial x^2} = a^2\cdot 2\sec^2(y + ax)\cdot\tan(y + ax) + a^2\cdot\frac{3}{4}(y - ax)^{-1/2}$ .... (1)

Also, $\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$

and $\frac{\partial^2 z}{\partial y^2} = 2\sec^2(y+ax)\cdot\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}$ .... (2)

From (1) and (2), we see that $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$.

Example 4 (a) : If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$ where $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$. **(M.U. 2016)**

Sol. : We have $\frac{\partial z}{\partial x} = \frac{e^x}{e^x + e^y}$ $\therefore$ $\frac{\partial^2 z}{\partial x^2} = \frac{(e^x+e^y)e^x - e^x(e^x)}{(e^x+e^y)^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$

$\frac{\partial z}{\partial y} = \frac{e^y}{e^x + e^y}$ $\therefore$ $\frac{\partial^2 z}{\partial y^2} = \frac{(e^x+e^y)e^y - e^y(e^y)}{(e^x+e^y)^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$

Now, $\frac{\partial^2 z}{\partial x \partial y} = e^x\left[-\frac{1}{(e^x+e^y)^2}\cdot e^y\right] = -\frac{e^{x+y}}{(e^x+e^y)^2}$

$$\therefore\quad rt - s^2 = \frac{e^{x+y}}{(e^x+e^y)^2}\cdot\frac{e^{x+y}}{(e^x+e^y)^2} - \left(-\frac{e^{x+y}}{(e^x+e^y)^2}\right)^2$$

$$= \left[\frac{e^{x+y}}{(e^x+e^y)^2}\right]^2 - \left[\frac{e^{x+y}}{(e^x+e^y)^2}\right]^2 = 0$$

Class (b) : 6 Marks

Example 1 (b) : If $u = e^{ax}\sin(x+bt)$ is the solution of $\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$ with the condition that $u \to 0$ as $x \to \infty$, find the values of a and b.

Sol. : We have, by differentiating partially w.r.t. t,

$$\frac{\partial u}{\partial t} = be^{ax}\cos(x+bt)$$

Now, differentiating partially w.r.t. x,

$$\frac{\partial u}{\partial x} = ae^{ax}\sin(x+bt) + e^{ax}\cos(x+bt)$$

Differentiating again w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = a^2e^{ax}\sin(x+bt) + 2ae^{ax}\cos(x+bt) - e^{ax}\sin(x+bt)$$

Putting these values in $\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$,

$$be^{ax}\cos(x+bt) = \mu a^2e^{ax}\sin(x+bt) + 2\mu ae^{ax}\cos(x+bt) - \mu e^{ax}\sin(x+bt)$$

$$\therefore\quad \mu(a^2-1)e^{ax}\sin(x+bt) + (2\mu a - b)e^{ax}\cos(x+bt) = 0$$

The equality will hold good only if the coefficients of sin $(x + bt)$ and cos $(x + bt)$ are equal to zero.

$\therefore$ Equating to zero the coefficients of sine and cosine,

$$\mu(a^2 - 1) = 0 \text{ and } 2\mu a - b = 0$$

$$\therefore \quad a^2 = 1 \quad i.e. \quad a = \pm 1 \text{ and } b = 2\mu a.$$

Since by data $u \to 0$ as $x \to \infty$, we get, from $u = e^{ax} \sin(x + bt)$, $a = -1$ $\therefore$ $b = -2\mu$.

[If $a = 1$, u does not tend to zero as $x \to \infty$. $\because$ $e^{-x} = \dfrac{1}{e^x} \to 0$ as $x \to \infty$ and $e^x \to \infty$ as $x \to \infty$.

Example 2 (b) : If $u = e^{x^2+y^2+z^2}$, prove that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = 8\, xyz\, u$.

Sol. : We have $\dfrac{\partial u}{\partial z} = e^{x^2+y^2+z^2} \cdot 2z$

$$\therefore \quad \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = 2z \cdot e^{x^2+y^2+z^2} \cdot 2y = 4yz \cdot e^{x^2+y^2+z^2}$$

$$\therefore \quad \frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = 4yz \cdot e^{x^2+y^2+z^2} \cdot 2x$$

$$= 8xyz \cdot e^{x^2+y^2+z^2} = 8xyzu.$$

Example 3 (b) : If $u = f\left(\dfrac{x^2}{y}\right)$, prove that

$$x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = 0 \text{ and } x^2\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x \partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = 0.$$ (M.U. 1994, 97, 99, 2004)

Sol. : We have $\dfrac{\partial u}{\partial x} = f'\left(\dfrac{x^2}{y}\right) \cdot \dfrac{2x}{y}$, $\dfrac{\partial u}{\partial y} = f'\left(\dfrac{x^2}{y}\right) \cdot \left(-\dfrac{x^2}{y^2}\right)$

$$\therefore \quad x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = f'\left(\frac{x^2}{y}\right)\left[\frac{2x^2}{y} - \frac{2x^2}{y}\right] = 0 \quad \text{.......... (1)}$$

Differentiating (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + 2y\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{.......... (2)}$$

Differentiating (1) partially w.r.t. y, now

$$x\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial u}{\partial y} + 2y\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{.......... (3)}$$

Multiply (2) by x, (3) by y and add,

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + 2xy\frac{\partial^2 u}{\partial x \partial y} + xy\frac{\partial^2 u}{\partial x \partial y} + 2y\frac{\partial u}{\partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = 0$$

But $x\dfrac{\partial u}{\partial x} + 2y\dfrac{\partial u}{\partial y} = 0$. Hence, $x^2\dfrac{\partial^2 u}{\partial x^2} + 3xy\dfrac{\partial^2 u}{\partial x \partial y} + 2y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

Example 4 (b) : If $u = \log (x^3 + y^3 + z^3 - 3xyz)$, prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$ **(M.U. 1999, 2002, 09)**

Sol. : We have $\text{l.h.s.} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$ **[Note this]**

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \qquad \text{.................... (1)}$$

Now, $\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3x^2 - 3yz)$

Similarly, $\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz}$, $\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$

$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 3\frac{x^2 + y^2 + z^2 - xy - yz - zx}{x^3 + y^3 + z^3 - 3xyz} = \frac{3}{(x+y+z)}$

[$\because (x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z) = x^3 + y^3 + z^3 - 3xyz.$ **(Note this)**]

Hence, from (1),

$$\text{l.h.s.} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\cdot\frac{3}{(x+y+z)}$$

$$= 3\left[\frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2}\right]$$

$$= -\frac{9}{(x+y+z)} = \text{r.h.s.}$$

Example 5 (b) : If $u = (1 - 2xy + y^2)^{-1/2}$, prove that

$$\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2\frac{\partial u}{\partial y}\right] = 0.$$ **(M.U. 1986, 88, 99, 2004, 05)**

Sol. : We have, $\text{l.h.s.} = -2x\frac{\partial u}{\partial x} + (1 - x^2)\frac{\partial^2 u}{\partial x^2} + 2y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ .... (1)

But as in the Ex. 4, page 6-10 above,

$\frac{\partial u}{\partial x} = u^3 y \qquad \therefore \frac{\partial^2 u}{\partial x^2} = 3u^2\frac{\partial u}{\partial x}\cdot y = 3u^5 y^2$

Also, $\frac{\partial u}{\partial y} = (x - y)u^3 \qquad \therefore \frac{\partial^2 u}{\partial y^2} = (x - y)\cdot 3u^2\frac{\partial u}{\partial y} - u^3 = (x - y)^2 3u^5 - u^3$

Putting these values in (1),

$\text{l.h.s.} = -2xyu^3 + (1 - x^2)\cdot 3u^5y^2 + 2y(x - y)u^3 + y^2(x - y)^2 3u^5 - u^3y^2$

$= 3u^5y^2[1 - x^2 + x^2 - 2xy + y^2] - 3u^3y^2$

$= 3u^5y^2(1 - 2xy + y^2) - 3u^3y^2.$

But by data $1 - 2xy + y^2 = u^{-2}$

$\therefore \text{l.h.s.} = 3u^5y^2u^{-2} - 3u^3y^2$

$= 3u^3y^2 - 3y^3y^2 = 0.$

Example 6 (b) : If $u = x^y$, show that $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$. (M.U. 2010, 1

Sol. : Since $u = x^y$, treating y constant $\dfrac{\partial u}{\partial x} = y\, x^{y-1}$ ....(1)

Treating x constant, $\dfrac{\partial u}{\partial y} = x^y \log x$ ....(2)

Differentiating (2) partially w.r.t. x, we get,

$$\frac{\partial^2 u}{\partial x \partial y} = x^y \cdot \frac{1}{x} + yx^{y-1}\log x = x^{y-1} + yx^{y-1}\log x$$

$$= x^{y-1}(1 + y\log x)$$

Differentiating again partially w.r.t. x, we get,

$$\frac{\partial^3 u}{\partial x^2 \partial y} = (y-1)\, x^{y-2} \cdot (1 + y\log x) + x^{y-1} \cdot \frac{y}{x}$$

$$\therefore \quad \frac{\partial^3 u}{\partial x^2 \partial y} = x^{y-2}[\,y - 1 + y(y-1)\log x + y\,]$$

$$= x^{y-2}[2y - 1 + y(y-1)\log x]$$

Now, differentiating (1) partially w.r.t. y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = x^{y-1} + yx^{y-1}\log x = x^{y-1}(1 + y\log x)$$

Differentiating again w.r.t. x, we get

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = (y-1)\, x^{y-2}(1 + y\log x) + x^{y-1} \cdot \frac{y}{x}$$

$$= x^{y-2}[\,y - 1 + y(y-1)\log x + y\,]$$

$$= x^{y-2}[2y - 1 + y(y-1)\log x]$$ ....(3)

Hence, from (2) and (3) the result follows.

Example 7 (b) : If $z = x^y + y^x$, verify that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$. (M.U. 1996, 2003, 04, 05)

Sol. : Differentiating z partially w.r.t. y, we get,

$$\frac{\partial z}{\partial y} = x^y \log x + xy^{x-1}$$

Differentiating this partially w.r.t. x, we get,

$$\frac{\partial^2 z}{\partial x \partial y} = yx^{y-1} \cdot \log x + x^y \cdot \frac{1}{x} + 1 \cdot y^{x-1} + xy^{x-1}\log y$$

$$= yx^{y-1} \cdot \log x + x^{y-1} + y^{x-1} + xy^{x-1}\log y$$ ....(1)

Now, differentiating z partially w.r.t. x, we get,

$$\frac{\partial z}{\partial x} = yx^{y-1} + y^x \log y$$

Differentiating this again partially w.r.t. y, we get,

$$\frac{\partial^2 z}{\partial y \partial x} = x^{y-1} + y \cdot x^{y-1} \log x + \frac{y^x}{y} + xy^{x-1} \log y$$

$$= yx^{y-1} \log x + x^{y-1} + y^{x-1} + xy^{x-1} \log y \quad \text{.................... (2)}$$

From (1) and (2), the result follows.

Example 8 (b) : If $u = f(r)$ and $r = \sqrt{x^2 + y^2}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$ **(M.U. 1993, 97)**

Sol. : Since $r^2 = x^2 + y^2 \quad \therefore \quad 2r \frac{\partial r}{\partial x} = 2x$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}. \quad \text{Similarly, } \frac{\partial r}{\partial y} = \frac{y}{r}$$

Now, $\frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = f'(r) \cdot \frac{x}{r}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \frac{x}{r} \cdot \frac{\partial r}{\partial x} + f'(r) \cdot \frac{1}{r} - f'(r) \cdot \frac{x}{r^2} \frac{\partial r}{\partial x}$$

Putting the value of $\frac{\partial r}{\partial x}$,

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \frac{x^2}{r^2} + f'(r) \cdot \frac{1}{r} - f'(r) \cdot \frac{x^2}{r^3}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \frac{y^2}{r^2} + f'(r) \cdot \frac{1}{r} - f'(r) \cdot \frac{y^2}{r^3}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \frac{(x^2 + y^2)}{r^2} + 2f'(r) \cdot \frac{1}{r} - f'(r) \cdot \frac{(x^2 + y^2)}{r^3}$$

$$= f''(r) + \frac{f'(r)}{r} \qquad [\because\ x^2 + y^2 = r^2]$$

Example 9 (b) : If $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r).$$ **(M.U. 1991, 93, 97, 2002)**

Sol. : Left to you.

Example 10 (b) : If $u = f(r^2)$ where $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 4r^2 f''(r^2) + 6f'(r^2).$$ **(M.U. 1992)**

Sol. : We have $2r \frac{\partial r}{\partial x} = 2x \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}.$ Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}.$

Now, since $u = f(r^2)$,

$$\frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = f'(r^2) \cdot 2r \cdot \frac{x}{r} = 2 \cdot f'(r^2) \cdot x$$

Similarly, $\frac{\partial u}{\partial y} = 2f'(r^2) \cdot y, \quad \frac{\partial u}{\partial z} = 2f'(r^2) \cdot z$

Now, $\frac{\partial^2 u}{\partial x^2} = 2 \cdot \left[f'(r^2) + x \cdot f''(r^2) \cdot 2r \cdot \frac{\partial r}{\partial x} \right] = 2\left[f'(r^2) + x \cdot f''(r^2) \cdot 2 \cdot r \cdot \frac{x}{r} \right]$

$\therefore \quad \frac{\partial^2 u}{\partial x^2} = 2f'(r^2) + 4f''(r^2) \cdot x^2$

Similarly, $\frac{\partial^2 u}{\partial y^2} = 2f'(r^2) + 4f''(r^2) \cdot y^2$ and $\frac{\partial^2 u}{\partial z^2} = 2f'(r^2) + 4f''(r^2) \cdot z^2$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6f'(r^2) + 4f''(r^2)[x^2 + y^2 + z^2]$$

$$= 6f'(r^2) + 4r^2 \cdot f''(r^2)$$

Example 11 (b) : If $u = r^m$, $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)\, r^{m-2}.$$ **(M.U. 1988, 95, 2001)**

Sol. : Since, $r^2 = x^2 + y^2 + z^2$, $\quad 2r\frac{\partial r}{\partial x} = 2x$

$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}$. Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\because \quad u = r^m$, $\frac{du}{dr} = m\, r^{m-1}$

Now, $\frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = m\, r^{m-1} \cdot \frac{x}{r} = m\, x\, r^{m-2}$

$\therefore \quad \frac{\partial^2 u}{\partial x^2} = m\, r^{m-2} + mx(m-2) \cdot r^{m-3}\, \frac{\partial r}{\partial x}$

$\therefore \quad \frac{\partial^2 u}{\partial x^2} = m\, r^{m-2} + m(m-2)\, r^{m-3} \cdot x \cdot \frac{x}{r}$

$$= m\, r^{m-2} + m(m-2)\, r^{m-4} \cdot x^2$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = m\, r^{m-2} + m(m-2)\, r^{m-4} \cdot y^2$

and $\frac{\partial^2 u}{\partial z^2} = m\, r^{m-2} + m(m-2)\, r^{m-4} \cdot z^2$

Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 3\, m\, r^{m-2} + m(m-2)\, r^{m-4}\, (x^2 + y^2 + z^2)$

$$= 3\, m\, r^{m-2} + m(m-2)\, r^{m-2}$$
$$= m(m+1)\, r^{m-2}.$$

Example 12 (b) : Show that $z = f(x + at) + \Phi\,(x - at)$ is a solution of

$a^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$ for all f and Φ (a, being constant). **(M.U. 1982, 91)**

Sol. : We have $z = f(x + at) + \Phi(x - at)$

$$\therefore \quad \frac{\partial z}{\partial x} = f'(x + at) + \Phi'(x - at)$$

and $$\frac{\partial^2 z}{\partial x^2} = f''(x + at) + \Phi''(x - at) \qquad \text{.......... (1)}$$

Further, $$\frac{\partial z}{\partial t} = af'(x + at) - a\Phi'(x - at)$$

and $$\frac{\partial^2 z}{\partial t^2} = a^2 f''(x + at) + a^2 \Phi''(x - at) \qquad \text{.......... (2)}$$

From (1) and (2), we get $a^2 \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial t^2}$ for all f and Φ. Hence, the required result.

Example 13 (b) : If $u = Ae^{-gx} \sin(nt - gx)$ satisfies the equation

$\dfrac{\partial u}{\partial t} = \mu \dfrac{\partial^2 u}{\partial x^2}$, prove that $g = \sqrt{\dfrac{n}{2\mu}}$. **(M.U. 1998, 2004, 07)**

[OR If $u = Ae^{-gx} \sin(nt - gx)$ satisfies the equation $\dfrac{\partial u}{\partial t} = \mu^2 \dfrac{\partial^2 u}{\partial x^2}$, prove that $g = \dfrac{1}{\mu}\sqrt{\dfrac{n}{2}}$.]

Sol. : We have

$$\frac{\partial u}{\partial x} = A[-ge^{-gx} \sin(nt - gx) - ge^{-gx} \cos(nt - gx)]$$

$$= -Age^{-gx}[\sin(nt - gx) + \cos(nt - gx)]$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = -Ag[-g \cdot e^{-gx}\{\sin(nt - gx) + \cos(nt - gx)\}$$
$$+ e^{-gx}\{-g\cos(nt - gx) + g\sin(nt - gx)\}]$$

$$= 2Ag^2 e^{-gx} \cos(nt - gx)$$

Further, $\dfrac{\partial u}{\partial t} = Ane^{-gx} \cos(nt - gx)$. But $\dfrac{\partial u}{\partial t} = \mu \dfrac{\partial^2 u}{\partial x^2}$ [By data]

$$\therefore \quad Ane^{-gx} \cos(nt - gx) = \mu \cdot 2A \cdot g^2 e^{-gx} \cos(nt - gx)$$

$$\therefore \quad n = 2\mu g^2 \qquad \therefore \quad g = \sqrt{\frac{n}{2\mu}}.$$

Example 14 (b) : If $u = (ar^n + br^{-n}) \cos(n\theta - \alpha)$ or

[$u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$], prove that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} = 0.$$ **(M.U. 1994, 96)**

Sol. : We have $\dfrac{\partial u}{\partial r} = (nar^{n-1} - nbr^{-n-1}) \cos(n\theta - \alpha)$

$$\therefore \quad \frac{\partial^2 u}{\partial r^2} = [n(n-1)ar^{n-2} + n(n+1)br^{-n-2}] \cos(n\theta - \alpha)$$

Further $\frac{\partial u}{\partial \theta} = (a\,r^n + b\,r^{-n})[-n\sin(n\theta - \alpha)]$

$\therefore \quad \frac{\partial^2 u}{\partial \theta^2} = (a\,r^n + b\,r^{-n})[-n^2\cos(n\theta - \alpha)]$

Putting these values in the l.h.s.

$$\therefore \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2}$$

$$= n(n-1)\,a\,r^{n-2}\cos(n\theta-\alpha) + n(n+1)\,b\,r^{-n-2}\cos(n\theta-\alpha) + n\,a\,r^{n-2}\cos(n\theta-\alpha) - n\,b\,r^{-n-2}\cos(n\theta-\alpha) - n^2\,a\,r^{n-2}\cos(n\theta-\alpha) - n^2\,b\,r^{-n-2}\cos(n\theta-\alpha)$$

$$= 0$$

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : If $z = u(x, y)\,e^{ax+by}$ where $u(x, y)$ is such that $\frac{\partial^2 u}{\partial x\,\partial y} = 0$, find the constants a, b such that $\frac{\partial^2 z}{\partial x\,\partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$.

Sol. : We have, from $z = u(x, y)\,e^{ax+by}$ ....(1)

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}\cdot e^{ax+by} + u\cdot e^{ax+by}\cdot a = e^{ax+by}\left(\frac{\partial u}{\partial x} + au\right) \qquad \text{..................(2)}$$

And $$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y}\cdot e^{ax+by} + u\cdot e^{ax+by}\cdot b = e^{ax+by}\left(\frac{\partial u}{\partial y} + bu\right) \qquad \text{..................(3)}$$

Differentiating (3) partially w.r.t. x

$$\frac{\partial^2 z}{\partial x\,\partial y} = e^{ax+by}\cdot a\cdot\left(\frac{\partial u}{\partial y} + bu\right) + e^{ax+by}\left(\frac{\partial^2 u}{\partial x\,\partial y} + b\cdot\frac{\partial u}{\partial x}\right) \qquad \text{..................(4)}$$

But since by data $\frac{\partial^2 u}{\partial x\,\partial y} = 0$, we get

$$\frac{\partial^2 z}{\partial x\,\partial y} = e^{ax+by}\left(a\cdot\frac{\partial u}{\partial y} + b\cdot\frac{\partial u}{\partial x} + abu\right) \qquad \text{..................(5)}$$

Further by data $$\frac{\partial^2 z}{\partial x\,\partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0 \qquad \text{..................(6)}$$

Putting the values from (1), (2), (3) and (5) in (6), we get,

$$e^{ax+by}\left[a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x} + abu - \frac{\partial u}{\partial x} - au - \frac{\partial u}{\partial y} - bu + u\right] = 0$$

$$\therefore \quad e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + au(b-1) - u(b-1)\right] = 0$$

$$\therefore \quad e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + (b-1)\cdot u(a-1)\right] = 0$$

Since $u \neq 0$, $\dfrac{\partial u}{\partial x} \neq 0$ and $\dfrac{\partial u}{\partial y} \neq 0$, we should have

$$a - 1 = 0, \quad b - 1 = 0 \quad i.e. \quad a = 1, \quad b = 1.$$

Example 2 (c) : If $u = e^{xyz} f\left(\dfrac{xy}{z}\right)$, prove that

$$x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = 2\,xy\,zu; \quad y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\,xy\,zu.$$

Hence, show that $x\dfrac{\partial^2 u}{\partial z \partial x} = y\dfrac{\partial^2 u}{\partial z \partial y}$. **(M.U. 1992, 99, 2018)**

Sol. : We have $$\frac{\partial u}{\partial x} = e^{xyz} \cdot yz \cdot f\left(\frac{xy}{z}\right) + e^{xyz} \cdot f'\left(\frac{xy}{z}\right) \cdot \frac{y}{z}$$

Similarly, $$\frac{\partial u}{\partial y} = e^{xyz} \cdot xz \cdot f\left(\frac{xy}{z}\right) + e^{xyz} \cdot f'\left(\frac{xy}{z}\right) \cdot \frac{x}{z}$$

and $$\frac{\partial u}{\partial z} = e^{xyz} \cdot xy \cdot f\left(\frac{xy}{z}\right) + e^{xyz} \cdot f'\left(\frac{xy}{z}\right) \cdot \left(-\frac{xy}{z^2}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = e^{xyz} \cdot xyz \cdot f\left(\frac{xy}{z}\right) + e^{xyz} \cdot f'\left(\frac{xy}{z}\right) \cdot \left(\frac{xy}{z}\right)$$
$$+ e^{xyz} \cdot xyz \cdot f\left(\frac{xy}{z}\right) + e^{xyz} \cdot f'\left(\frac{xy}{z}\right) \cdot \left(-\frac{xy}{z}\right)$$
$$= 2e^{xyz} \cdot xyz \cdot f\left(\frac{xy}{z}\right) = 2\,xy\,zu.$$

Similarly, it can be easily proved that $y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2\,xy\,zu$

Now, differentiating both sides of $x\dfrac{\partial u}{\partial x} + z\dfrac{\partial u}{\partial z} = 2\,xy\,zu$ partially w.r.t. z,

$$x\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial u}{\partial z} + z\frac{\partial^2 u}{\partial z^2} = 2\,xyu + 2\,xyz\frac{\partial u}{\partial z} \quad \text{.................... (1)}$$

Further differentiating both sides of $y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2\,xy\,zu$ partially w.r.t. z

$$y\frac{\partial^2 u}{\partial z \partial y} + \frac{\partial u}{\partial z} + z\frac{\partial^2 u}{\partial z^2} = 2\,xy\,u + 2\,xyz\frac{\partial u}{\partial z} \quad \text{.................... (2)}$$

From (1) and (2) it is clear that $x\dfrac{\partial^2 u}{\partial z \partial x} = y\dfrac{\partial^2 u}{\partial z \partial y}$.

Example 3 (c) : If $z = x \log(x + r) - r$ where $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x + r}, \quad \frac{\partial^3 z}{\partial x^3} = -\frac{x}{r^3}.$$ **(M.U. 1983, 91, 2002, 04, 08, 09)**

Sol. : Since $r^2 = x^2 + y^2$ as seen before $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ and $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

Differentiating $z = x \log(x + r) - r$ partially w.r.t. x,

$$\frac{\partial z}{\partial x} = \left[\frac{x}{x+r}\left(1+\frac{\partial r}{\partial x}\right) + \log(x+r)\cdot 1\right] - \frac{\partial r}{\partial x}$$

$$= \left[\frac{x}{x+r}\left(1+\frac{x}{r}\right) + \log(x+r)\right] - \frac{x}{r}$$

$$= \frac{x}{r} + \log(x+r) - \frac{x}{r} = \log(x+r)$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = \frac{1}{x+r}\left(1+\frac{\partial r}{\partial x}\right) = \frac{1}{x+r}\left(1+\frac{x}{r}\right) = \frac{1}{r} \qquad \text{.......(1)}$$

Now, differentiating $z = x\log(x+r) - r$ partially w.r.t. y,

$$\frac{\partial z}{\partial y} = x \cdot \frac{1}{x+r}\left(\frac{\partial r}{\partial y}\right) - \frac{\partial r}{\partial y} = \frac{x}{x+r}\cdot\frac{y}{r} - \frac{y}{r}$$

$$= \frac{y}{r}\left(\frac{x}{x+r} - 1\right) = -\frac{y}{x+r}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{(x+r)(1) - y(\partial r/\partial y)}{(x+r)^2} = -\frac{(x+r) - y\cdot(y/r)}{(x+r)^2}$$

$$= -\frac{rx + r^2 - y^2}{r(x+r)^2} = -\frac{rx + x^2}{r(x+r)^2} \qquad [\because r^2 - y^2 = x^2]$$

$$\therefore \quad \frac{\partial^2 z}{\partial y^2} = -\frac{x(r+x)}{r(x+r)^2} = -\frac{x}{r(x+r)} \qquad \text{.......(2)}$$

From (1) and (2),

$$\frac{\partial^2 z}{\partial z^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{r} - \frac{x}{r(x+r)} = \frac{x+r-x}{r(x+r)} = \frac{1}{x+r}$$

Now from (1), $\dfrac{\partial^3 z}{\partial x^3} = -\dfrac{1}{r^2}\cdot\dfrac{\partial r}{\partial x} = -\dfrac{1}{r^2}\cdot\dfrac{x}{r} = -\dfrac{x}{r^3}$.

Example 4 (c) : If $x = e^{r\cos\theta}\cos(r\sin\theta)$, $y = e^{r\cos\theta}\sin(r\sin\theta)$,

prove that $\dfrac{\partial x}{\partial r} = \dfrac{1}{r}\cdot\dfrac{\partial y}{\partial \theta}$, $\dfrac{\partial y}{\partial r} = -\dfrac{1}{r}\cdot\dfrac{\partial x}{\partial \theta}$. (M.U. 2004, 06)

Hence, deduce that $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r}\cdot\dfrac{\partial x}{\partial r} + \dfrac{1}{r^2}\cdot\dfrac{\partial^2 x}{\partial \theta^2} = 0.$ (M.U. 1999)

Sol. : **Since** $x = e^{r\cos\theta}\cos(r\sin\theta)$,

$$\frac{\partial x}{\partial r} = e^{r\cos\theta}\cdot\cos\theta\cos(r\sin\theta) - e^{r\cos\theta}\cdot\sin(r\sin\theta)\sin\theta$$

$$= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\cos(r\sin\theta + \theta) \qquad \text{.......(i)}$$

And $$\frac{\partial x}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta)\cos(r\sin\theta) + e^{r\cos\theta}[-\sin(r\sin\theta)][r\cos\theta]$$

$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= -re^{r\cos\theta}\sin(r\sin\theta + \theta) \qquad \text{.......(ii)}$$

Similarly, $\dfrac{\partial y}{\partial r} = e^{r\cos\theta}\sin(r\sin\theta + \theta)$ .... (iii)

and $\dfrac{\partial y}{\partial \theta} = r\,e^{r\cos\theta}\cos(r\sin\theta + \theta)$ .... (iv)

From (i) and (iv), we get $\dfrac{\partial x}{\partial r} = \dfrac{1}{r}\cdot\dfrac{\partial y}{\partial \theta}$ .... (v)

From (ii) and (iii), we get $\dfrac{\partial y}{\partial r} = -\dfrac{1}{r}\cdot\dfrac{\partial x}{\partial \theta}$ .... (vi)

Now, differentiating (v) w.r.t. r, we get

$$\frac{\partial^2 x}{\partial r^2} = -\frac{1}{r^2}\cdot\frac{\partial y}{\partial \theta} + \frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\,\partial \theta} \quad \text{..................... (vii)}$$

From (vi), we get $\dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}$

Differentiating this w.r.t. θ, we get

$$\frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial r\,\partial \theta} \qquad \therefore \quad \frac{1}{r^2}\cdot\frac{\partial^2 x}{\partial \theta^2} = -\frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\,\partial \theta} \quad \text{..................... (viii)}$$

Adding (vii) and (viii), $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r^2}\cdot\dfrac{\partial^2 x}{\partial \theta^2} = -\dfrac{1}{r^2}\cdot\dfrac{\partial y}{\partial \theta}$ .... (ix)

But from (v), $\dfrac{1}{r^2}\cdot\dfrac{\partial y}{\partial \theta} = \dfrac{1}{r}\cdot\dfrac{\partial x}{\partial r}$

Hence, (ix) becomes $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r}\cdot\dfrac{\partial x}{\partial r} + \dfrac{1}{r^2}\cdot\dfrac{\partial^2 x}{\partial \theta^2} = 0.$

Type III : Examples Satisfying Laplace Equation : Class (b) : 6 marks

Example 1 (b) : If $u = \cos 4x \cos 3y \sin h\, 5z$, prove that u satisfies Laplace equation *i.e.* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

Sol. : We have

$$\frac{\partial u}{\partial x} = -4\cdot\sin 4x\cos 3y\sin h\,5z \qquad \therefore \quad \frac{\partial^2 u}{\partial x^2} = -16\cdot\cos 4x\cos 3y\sin h5z = -16\,u$$

Similarly, $\dfrac{\partial u}{\partial y} = -3\cdot\cos 4x\sin 3y\sin h\,5z \qquad \therefore \quad \dfrac{\partial^2 u}{\partial y^2} = -9\cdot\cos 4x\cos 3y\sin h\,5z = -9\,u$

And $\dfrac{\partial u}{\partial z} = 5\cdot\cos 4x\cos 3y\cos h\,5z \qquad \therefore \quad \dfrac{\partial^2 u}{\partial z^2} = 25\cdot\cos 4x\cos 3y\sin h\,5z = 25\,u$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -16\,u - 9\,u + 25\,u = 0.$$

Example 2 (b) : If $u = \frac{1}{r}$, $r = \sqrt{x^2 + y^2 + z^2}$, [Or if $u = (x^2 + y^2 + z^2)^{-1/2}$] prove that u statisfi

Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

(M.U. 199

Sol. : We have $\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = -\frac{1}{r^2} \cdot \frac{x}{r} = -\frac{x}{r^3}$$

and $$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{r^3} + \frac{3x}{r^4} \cdot \frac{\partial r}{\partial x} = -\frac{1}{r^3} + \frac{3x^2}{r^5}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = -\frac{1}{r^3} + \frac{3y^2}{r^5}$ and $\frac{\partial^2 u}{\partial z^2} = -\frac{1}{r^3} + \frac{3z^2}{r^5}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

EXERCISE – III