MAXIMA AND MINIMA

Friday, January 14, 2022 1:47 PM

WORKING RULE TO FIND EXTREMUM VALUES:

$z = f(x, y)$

(1) $\left.\begin{aligned} \frac{\partial f}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} &= 0 \end{aligned}\right\}$ Solve them to get values of (x, y) which satisfy both equations

These points are called the critical or the stationary points

(2) $r = \frac{\partial^2 f}{\partial x^2}$, $t = \frac{\partial^2 f}{\partial y^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$

find r, t, s at every critical point.

Necessary condition : $rt - s^2 > 0$

If $rt - s^2 \leq 0$ then ignore that critical point (neither max nor min

if $rt - s^2 > 0$ and r (or t) > 0

Then it is point of minima

If $rt - s^2 > 0$ and r (or t) < 0

then it is point of maxima.

one variable function

$y = f(x)$

(1) $f'(x) = 0$

Solving we get values of x

critical points

Stationary points

let $x = a, b$ are critical points

(2) find $f''(x)$

find $f''(x)$ at critical points

$f''(a) > 0 \rightarrow$ minima

$f''(b) < 0 \rightarrow$ maxima

SOME SOLVED EXAMPLES:

1. Discuss the maxima and minima of the function $x^2 + y^2 + 6x + 12$

Soln :- $f(x, y) = x^2 + y^2 + 6x + 12$

Step-I :- $\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x + 6 \\ \frac{\partial f}{\partial y} &= 2y \end{aligned}\right\}$ $\frac{\partial f}{\partial x} = 0 \Rightarrow 2x + 6 = 0 \Rightarrow x = -3$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y = 0 \Rightarrow y = 0.$

$\therefore (-3, 0)$ is the stationary point.

Step-II :- $r = \frac{\partial^2 f}{\partial x^2} = 2$, $s = \frac{\partial^2 f}{\partial x \partial y} = 0$, $t = \frac{\partial^2 f}{\partial y^2} = 2$

$\partial x \partial y$ $\qquad \partial y^2$

Step-III :- at $(-3,0)$ $\quad r=2,\ s=0,\ t=2$

$\therefore rt-s^2 = 4 > 0$

and $r = 2 > 0$

Hence $f(x,y)$ is minimum at $(-3,0)$

$f_{min} = f(-3,0) = (-3)^2 + (0)^2 + 6(-3) + 12$

$= 9 - 18 + 12 = 3$

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2. Find the extreme values of the function $x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$, if any.

Sol^n :- $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$.

Step-I :- For extreme values, $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 3y^2 - 6x = 0 \Rightarrow x^2 + y^2 - 2x = 0$ — (1)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 6xy - 6y = 0 \Rightarrow xy - y = 0 \Rightarrow (x-1)y = 0$

$\Rightarrow x - 1 = 0$ or $y = 0$

$\Rightarrow x = 1$ or $y = 0$

Using $x = 1$ in eq^n (1)

$1 + y^2 - 2 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

$\therefore$ stationary points are $(1,1)$ and $(1,-1)$

Using $y=0$ in eq^n (1)

$x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$

$\therefore$ stationary points are $(0,0)$ and $(2,0)$

Step-II :- $r = \frac{\partial^2 f}{\partial x^2} = 6x - 6$

$s = \frac{\partial^2 f}{\partial x \partial x} = 6y$ and $t = \frac{\partial^2 f}{\partial y^2} = 6x - 6$.

Step-III :-

(x,y)	r	s	t	$rt-s^2$	sign of r	conclusion
$(0,0)$	-6	0	-6	$36 > 0$	$r = -6 < 0$	Maximum
$(2,0)$	6	0	6	$36 > 0$	$r = 6 > 0$	Minimum

$(0,0)$	-6	0	-6	$36>0$	$r=-6<0$	Maximum
$(2,0)$	6	0	6	$36>0$	$r=6>0$	Minimum
$(1,1)$	0	6	0	$-36<0$	—	neither max nor minimum
$(1,-1)$	0	-6	0	$-36<0$	—	neither max nor min.

Hence $f(x,y)$ is maximum at $(0,0)$ and minimum at $(2,0)$

$$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$$

$$f_{max} = f(0,0) = 7$$

$$f_{min} = f(2,0) = (2)^3 - 3(2)^2 + 7 = 8 - 12 + 7 = 3$$

3. Find the extreme value of $xy(a-x-y)$

Soln:- $f(x,y) = xy(a-x-y) = axy - x^2y - xy^2$

Step-I :- For extreme values, $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 0 \Rightarrow ay - 2xy - y^2 = 0$

$y(a - 2x - y) = 0 \Rightarrow y = 0$ or $a - 2x - y = 0$ — (1)

$\frac{\partial f}{\partial y} = 0 \Rightarrow ax - x^2 - 2xy = 0$

$x(a - x - 2y) = 0 \Rightarrow x = 0$ or $a - x - 2y = 0$ — (2)

considering four pairs of equations from (1) & (2)

1^{st} eqn	2^{nd} eqn	stationary point
$y=0$	$x=0$	$(0,0)$
$y=0$	$a-x-2y=0$	$(a,0)$
$a-2x-y=0$	$x=0$	$(0,a)$
$a-2x-y=0$, $2x+y=a$	$a-x-2y=0$, $x+2y=a$	$\left(\frac{a}{3},\frac{a}{3}\right)$

Step-II :- $r = \frac{\partial^2 f}{\partial x^2} = -2y \qquad s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y$

$t = \frac{\partial^2 f}{\partial y^2} = -2x$

Step-III :-

(x,y)	r	s	t	$rt-s^2$	sign of r	conclusion

(x,y)	r	s	t	$rt-s^2$	sign of r	conclusion
$(0,0)$	0	a	0	$-a^2<0$	—	Neither max nor min
$(a,0)$	0	$-a$	$-2a$	$-a^2<0$	—	Neither max nor min
$(0,a)$	$-2a$	a	0	$-a^2<0$	—	Neither max nor min
$\left(\frac{a}{3},\frac{a}{3}\right)$	$-\frac{2a}{3}$	$-\frac{a}{3}$	$-\frac{2a}{3}$	$\frac{a^2}{3}>0$	$r=-\frac{2a}{3}$	Max or Min

Hence $f(x,y)$ is maximum or minimum at $\left(\frac{a}{3},\frac{a}{3}\right)$ depending on whether $a>0$ or $a<0$

$$f_{extremum} = \left(\frac{a}{3}\right)\left(\frac{a}{3}\right)\left(a-\frac{2}{3}\right) = \frac{a^3}{27}$$

$f(x,y) = xy(a-x-y)$

4. Examine the function $x^3y^2(12-3x-4y)$ for extreme values.

Soln:- $f(x,y) = x^3y^2(12-3x-4y) = 12x^3y^2 - 3x^4y^2 - 4x^3y^3$

$$\frac{\partial f}{\partial x} = 36x^2y^2 - 12x^3y^2 - 12x^2y^3$$

$$\frac{\partial f}{\partial y} = 24x^3y - 6x^4y - 12x^3y^2$$

If $\frac{\partial f}{\partial x} = 0 \Rightarrow 36x^2y^2 - 12x^3y^2 - 12x^2y^3 = 0$

$\Rightarrow 3x^2y^2 - x^3y^2 - x^2y^3 = 0$

$\Rightarrow x^2y^2(3-x-y) = 0$

$\Rightarrow x=0$ or $y=0$ or $3-x-y=0$ — ①

ie $x+y=3$

If $\frac{\partial f}{\partial y} = 0 \Rightarrow 24x^3y - 6x^4y - 12x^3y^2 = 0$

$\Rightarrow 4x^3y - x^4y - 2x^3y^2 = 0$

$\Rightarrow x^3y(4-x-2y) = 0$

$\Rightarrow x=0$ or $y=0$ or $4-x-2y=0$

ie $x+2y=4$ ②

Considering six pairs of equations from ① & ②

$x=0$	$y=0$	$(0,0)$
$x=0$	$x+2y=4$	$(0,2)$

$y=0$	$x+2y=4$	$(4,0)$
$x+y=3$	$x=0$	$(0,3)$
$x+y=3$	$y=0$	$(3,0)$
$x+y=3$	$x+2y=4$	$(2,1)$

$\therefore$ we get six stationary points as above.

Step-II $\therefore$ $r = \dfrac{\partial^2 f}{\partial x^2} = 72xy^2 - 36x^2y^2 - 24xy^3$

$s = \dfrac{\partial^2 f}{\partial x \partial y} = 72x^2y - 24x^3y - 36x^2y^2$

$t = \dfrac{\partial^2 f}{\partial y^2} = 24x^3 - 6x^4 - 24x^3y$

Step-III :-

(x,y)	r	s	t	$rt-s^2$	sign of r	conclusion
$(0,0)$	0	0	0	0	-	No conclusion
$(0,2)$	0	0	0	0	-	"
$(4,0)$	0	0	0	0	-	"
$(0,3)$	0	0	0	0	-	"
$(3,0)$	0	0	162	0	-	"
$(2,1)$	-48	-48	-96	2304>0	$r<0$	Maximum

Hence $f(x,y)$ is maximum at $(2,1)$

$f_{max} = f(2,1) = (2)^3(1)^2(12-6-4) = 16.$

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5. Find the extreme values of $\sin x + \sin y + \sin(x+y)$

Soln :- $f(x,y) = \sin x + \sin y + \sin(x+y)$

Step-1 :- For extreme values $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$.

$\dfrac{\partial f}{\partial x} = \cos x + \cos(x+y)$ and $\dfrac{\partial f}{\partial y} = \cos y + \cos(x+y)$

$\dfrac{\partial f}{\partial x} = 0 \Rightarrow \cos x + \cos(x+y) = 0$ ——— (1)

$\dfrac{\partial f}{\partial y} = 0 \Rightarrow \cos y + \cos(x+y) = 0$ ——— (2)

equating (1) & (2)

$$\cos x + \cos(x+y) = \cos y + \cos(x+y)$$

$$\Rightarrow \cos x = \cos y$$

$$\Rightarrow \boxed{x = y}$$

Using $x = y$ in eqn (1)

$$\cos x + \cos 2x = 0$$

$$\cos x = -\cos 2x$$

$$= \cos(\pi + 2x) \text{ or } \cos(\pi - 2x)$$

$$\Rightarrow x = \pi + 2x \quad \text{or} \quad x = \pi - 2x$$

$$\Rightarrow x = -\pi \quad \text{or} \quad x = \frac{\pi}{3}$$

$$\Rightarrow y = -\pi \quad \text{or} \quad y = \frac{\pi}{3}$$

$\therefore$ There are 2 stationary points $(-\pi, \pi)$ and $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

Step-2 :- $r = \dfrac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$

$s = \dfrac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$t = \dfrac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

$r = -\sin\frac{\pi}{3} - \sin 2\frac{\pi}{3}$

$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$

$= -\sqrt{3}$

Step-3 :-

(x, y)	r	s	t	$rt - s^2$	sign of r	Conclusion
$(-\pi, \pi)$	0	0	0	0	—	No conclusion
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4} > 0$	$r = -\sqrt{3} < 0$	Maximum

$\therefore f(x,y) = \sin x + \sin y + \sin(x+y)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $f_{max} = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{2}$

6. Find the extreme values of $\sin x \sin y \sin(x + y)$

Soln :- $f(x,y) = \sin x \sin y \sin(x+y)$

Step-I :- $\dfrac{\partial f}{\partial x} = \sin y \left[\sin x \cos(x+y) + \cos x \sin(x+y)\right]$

$= \sin y \sin(2x+y)$

$\dfrac{\partial f}{\partial y} = \sin x \left[\sin y \cos(x+y) + \cos y \sin(x+y)\right]$

$$= \sin x \sin(x+2y)$$

$$\frac{\partial f}{\partial x} = 0 \implies \sin y \sin(2x+y) = 0$$

$$\implies \sin y = 0 \quad \text{or} \quad \sin(2x+y) = 0$$

$$\implies y = 0 \text{ or } \underline{y = \pi} \quad \text{or} \quad \underline{2x+y=0} \quad \text{or} \quad \underline{2x+y=\pi} \quad (1)$$

$$\frac{\partial f}{\partial y} = 0 \implies \sin x \sin(x+2y) = 0$$

$$\implies \sin x = 0 \quad \text{or} \quad \sin(x+2y) = 0$$

$$\implies \underline{x=0} \text{ or } \underline{x=\pi} \quad \text{or} \quad \underline{x+2y=0} \quad \text{or} \quad \underline{x+2y=\pi} \quad (2)$$

$y=0,\ x=0 \implies (0,0)$

$y=0,\ x=\pi \implies (\pi,0)$

$y=\pi,\ x=0 \implies (0,\pi)$

$y=\pi,\ x=\pi \implies (\pi,\pi)$

$2x+y=\pi,\ x+2y=\pi \implies \left(\frac{\pi}{3},\frac{\pi}{3}\right)$

These are 5 stationary points.

$2\sin\frac{\pi}{3}\cos\pi$

$2\left(\frac{\sqrt3}{2}\right)(-1)$

$= -\sqrt3$

Step-2 :- $r = \frac{\partial^2 f}{\partial x^2} = 2\sin y \cos(2x+y)$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \cos y \sin(2x+y) + \sin y \cos(2x+y)$$
$$= \sin(2x+2y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2\sin x \cos(x+2y)$$

Step-3

(x,y)	r	s	t	$rt-s^2$	sign of r	conclusion
$(0,0)$	0	0	0	0	$-$	No conclusion
$(0,\pi)$	0	0	0	0	$-$	No conclusion
$(\pi,0)$	0	0	0	0	$-$	No conclusion
(π,π)	0	0	0	0	$-$	No conclusion
$\left(\frac{\pi}{3},\frac{\pi}{3}\right)$	$-\sqrt3$	$-\frac{\sqrt3}{2}$	$-\sqrt3$	$\frac{9}{4}>0$	$r=-\sqrt3<0$	Maximum

$f(x,y) = \sin x \sin y \sin(x+y)$ is maximum at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

and $f_{max} = \left(\frac{\sqrt3}{2}\right)\left(\frac{\sqrt3}{2}\right)\left(\frac{\sqrt3}{2}\right) = \frac{3\sqrt3}{8}$

7. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.

and $f_{max} = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{8}$

7. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.

Soln:- Let x, y, z be the parts of 120

$\therefore x+y+z = 120 \Rightarrow z = 120-x-y$

$f = xy+yz+zx$

$= xy + y(120-x-y) + (120-x-y)x$

$f(x,y) = xy + 120y - xy - y^2 + 120x - x^2 - xy$

$f(x,y) = 120(x+y) - x^2 - y^2 - xy$

Step-1 :- $\frac{\partial f}{\partial x} = 120 - 2x - y$

$\frac{\partial f}{\partial y} = 120 - 2y - x$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 120-2x-y = 0 \Rightarrow 2x+y=120$ — (1)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 120-2y-x = 0 \Rightarrow x+2y=120$ — (2)

Solve (1) & (2), we get

$x = 40, \; y = 40$

$\therefore$ stationary point is $(40, 40)$

Step-2 :- $r = \frac{\partial^2 f}{\partial x^2} = -2, \; s = \frac{\partial^2 f}{\partial x \partial y} = -1, \; t = \frac{\partial^2 f}{\partial y^2} = -2$

Step-3 at $(40,40)$

$r = -2, \; s = -1, \; t = -2$

$rt - s^2 = 3 > 0$ & $r = -2 < 0$

$\therefore$ $(40,40)$ is the point is maxima.

$z = 120 - x - y = 120 - 80 = 40$

$\therefore$ 120 should be divided as 40, 40, 40 so get the maximum value of the function.

$f_{max} = xy + yz + zx = 1600 + 1600 + 1600 = 4800$

8. Find the points on the surface $z^2 = xy + 1$nearest to the origin. Also find that distance.

Sol^n:- Let $P(x, y, z)$ be any point on the surface

$$z^2 = xy+1$$

It's distance from the origin is

$$d = \sqrt{x^2+y^2+z^2} \Rightarrow d^2 = x^2+y^2+z^2$$
$$= x^2+y^2+xy+1$$

H.W

$\therefore f(x,y) = x^2+y^2+xy+1$

Step-1 :- $\frac{\partial f}{\partial x} = 2x+y$, $\frac{\partial f}{\partial y} = 2y+x$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} = 0 &\Rightarrow 2x+y=0 \\ \frac{\partial f}{\partial y} = 0 &\Rightarrow 2y+x=0 \end{aligned}\right\} \Rightarrow x=0,\ y=0$$

$(0,0)$ is the stationary point.

Step-2 :- $r = \frac{\partial^2 f}{\partial x^2} = 2$, $s = \frac{\partial^2 f}{\partial x \partial y} = 1$, $t = \frac{\partial^2 f}{\partial y^2} = 2$

$\therefore$ at $(0,0)$

$r=2,\ s=1,\ t=2$

$rt - s^2 = 3 > 0$ and $r = 2 > 0$

$\therefore$ $f(x,y)$ is minimum at $(0,0)$.

Now $z^2 = xy+1 = 1 \Rightarrow z = \pm 1$

$\therefore$ The points on the surface nearest to the origin are $(0,0,1)$ and $(0,0,-1)$

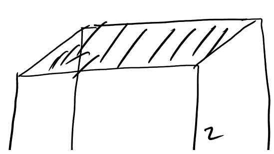
and min distance $= \sqrt{x^2+y^2+z^2} = 1$ unit.

9. A rectangular box open at the top is to have a volume of 108 cubic meters. Find the dimensions of the box if its total surface area is minimum.

Sol^n:- Let x, y, z be the dimensions of the box.

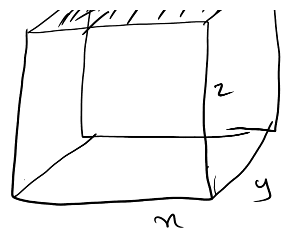
Let V be the volume & S be the surface area of the box.

$\therefore V = xyz = 108$



$\therefore$

$\therefore V = xyz = 108$



also $S = xy + 2yz + 2zx$

$$= xy + 2y\left(\frac{108}{xy}\right) + 2\left(\frac{108}{xy}\right)x$$

$$f(x,y) = xy + 216\left(\frac{1}{x} + \frac{1}{y}\right)$$

Step-1 : $\frac{\partial f}{\partial x} = y - \frac{216}{x^2}$, $\frac{\partial f}{\partial y} = x - \frac{216}{y^2}$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow y - \frac{216}{x^2} = 0 \quad \text{—— (1)}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow x - \frac{216}{y^2} = 0 \quad \text{—— (2)}$$

$$(1) \Rightarrow \quad y = \frac{216}{x^2}$$

Sub in (2) $\quad x - \frac{216}{\left(\frac{216}{x^2}\right)^2} = 0$

$$x - \frac{x^4}{216} = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x^3 = 216$$

but $x \neq 0$ as it is dimension

$\therefore x^3 = 216 \Rightarrow x = 6$

$y = \frac{216}{x^2} \Rightarrow y = 6$

$\therefore (6,6)$ is the stationary point

Step-2 : $r = \frac{\partial^2 f}{\partial x^2} = \frac{432}{x^3}$, $s = \frac{\partial^2 f}{\partial x \partial y} = 1$, $t = \frac{432}{y^3}$

at $(6,6)$, $r = 2$, $s = 1$, $t = 2$

$\therefore rt - s^2 = 3 > 0$ and $r > 0$

$\therefore f(x,y)$ is minimum at $(6,6)$

$$\therefore z = \frac{108}{xy} = \frac{108}{36} = 3$$

$\therefore z = \frac{108}{xy} = \frac{108}{36} = 3$

$\therefore$ The dimension of the box should be 6m, 6m, 3m to get min surface area.

min surface area $= xy + 2yz + 2zx = 36 + 36 + 36$
$= 108$ sq.m.

10. Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Solution: Let x, y, z be the length, breadth and height of the rectangular solid and V be its volume.

$V = x\,y\,z$ ……….. (1)

Let the given sphere be $x^2 + y^2 + z^2 = a^2$, $z^2 = a^2 - x^2 - y^2$

Substituting in Eq (1) $V = xy\sqrt{a^2 - x^2 - y^2}$

$V^2 = x^2y^2(a^2 - x^2 - y^2)$

Let $f(x, y) = V^2 = x^2y^2(a^2 - x^2 - y^2)$ …………. (2)

Step I: For extreme values, $\frac{\partial f}{\partial x} = 0$

$y^2[2x(a^2 - x^2 - y^2) + x^2(-2x)] = 0$

$2xy^2(a^2 - 2x^2 - y^2) = 0$

$x = 0, y = 0, 2x^2 + y^2 = a^2$ …………… (3)

$\frac{\partial f}{\partial y} = 0$

$x^2[2y(a^2 - x^2 - y^2) + y^2(-2y)] = 0$

$2x^2y(a^2 - x^2 - 2y^2) = 0$

$x = 0, y = 0, x^2 + 2y^2 = a^2$ …………… (4)

But x and y are the sides of the rectangular solid, and therefore, cannot be zero. Solving

$2x^2 + y^2 = a^2$ and $x^2 + 2y^2 = a^2$

$x^2 = \frac{a^2}{3}, y^2 = \frac{a^2}{3}, x = \frac{a}{\sqrt{3}}, y = \frac{a}{\sqrt{3}}$ [$\because$ side cannot be negative]

$z = \sqrt{a^2 - \frac{a^2}{3} - \frac{a^2}{3}} = \frac{a}{\sqrt{3}}$ Stationary points are $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

Step II:

$r = \frac{\partial^2 f}{\partial x^2} = 2a^2y^2 - 12x^2y^2 - 2y^4$

$s = \frac{\partial^2 f}{\partial x\,\partial y} = 4a^2xy - 8x^3y - 8xy^3$

$t = \frac{\partial^2 f}{\partial y^2} = 2a^2x^2 - 2x^4 - 12x^2y^2$

Step III: At $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right), r = \frac{2a^4}{3} - \frac{4a^4}{3} - \frac{2a^4}{9} = -\frac{8a^4}{9}$

$s = \frac{4a^4}{3} - \frac{8a^4}{9} - \frac{8a^4}{9} = -\frac{4a^4}{9}$

$t = \frac{2a^4}{3} - \frac{2a^4}{9} - \frac{12a^4}{9} = -\frac{8a^4}{9}$

$rt - s^2 = \frac{64a^8}{81} - \frac{16a^8}{81} = \frac{48a^8}{81} > 0; rt - s^2 > 0$ and $r < 0$

$f(x, y)$ i.e, V^2 is maximum at $x = y = z$ and hence, V is maximum when $x = y = z$, i.e the rectangular solid is a cube