1. If $u = \cos(\sqrt{x} + \sqrt{y})$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}(\sqrt{x} + \sqrt{y})\sin(\sqrt{x} + \sqrt{y}) = 0$.

Soln : $u = \cos(\sqrt{x} + \sqrt{y})$

differentiate u partially wrt x

$\frac{\partial u}{\partial x} = -\sin(\sqrt{x} + \sqrt{y}) \cdot \frac{\partial}{\partial x}(\sqrt{x} + \sqrt{y})$

$= -\sin(\sqrt{x} + \sqrt{y})(\frac{1}{2\sqrt{x}} + 0) = \frac{-1}{2\sqrt{x}}\sin(\sqrt{x} + \sqrt{y})$

differentiate u partially wrt y

$\frac{\partial u}{\partial y} = -\sin(\sqrt{x} + \sqrt{y}) \cdot \frac{\partial}{\partial y}(\sqrt{x} + \sqrt{y})$

$= -\sin(\sqrt{x} + \sqrt{y})(0 + \frac{1}{2\sqrt{y}}) = \frac{-1}{2\sqrt{y}}\sin(\sqrt{x} + \sqrt{y})$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left(\frac{-1}{2\sqrt{x}}\sin(\sqrt{x} + \sqrt{y})\right) + y\left(\frac{-1}{2\sqrt{y}}\sin(\sqrt{x} + \sqrt{y})\right)$

$= -\frac{\sqrt{x}}{2}\sin(\sqrt{x} + \sqrt{y}) - \frac{\sqrt{y}}{2}\sin(\sqrt{x} + \sqrt{y})$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\sin(\sqrt{x} + \sqrt{y})\left[\sqrt{x} + \sqrt{y}\right]$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}(\sqrt{x} + \sqrt{y})\sin(\sqrt{x} + \sqrt{y}) = 0$.

Hence proved.

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

2. If $z(x + y) = x^2 + y^2$, prove that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

Soln :- $z = \frac{x^2 + y^2}{x + y}$

differentiating z, partially wrt x

$\frac{\partial z}{\partial x} = \frac{(x+y) \cdot \frac{\partial}{\partial x}(x^2 + y^2) - (x^2 + y^2) \cdot \frac{\partial}{\partial x}(x+y)}{(x+y)^2}$

$$= \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

differentiating z partially wrt y

$$\frac{\partial z}{\partial y} = \frac{(x+y)\frac{\partial}{\partial y}(x^2+y^2) - (x^2+y^2)\cdot\frac{\partial}{\partial y}(x+y)}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)(2y) - (x^2+y^2)(1)}{(x+y)^2} = \frac{2xy + y^2 - x^2}{(x+y)^2}$$

$$LHS = \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{2xy + y^2 - x^2}{(x+y)^2}\right]^2$$

$$= \left[\frac{2(x^2 - y^2)}{(x+y)^2}\right]^2 = \left[\frac{2(x+y)(x-y)}{(x+y)^2}\right]^2$$

$$\therefore LHS = 4\frac{(x-y)^2}{(x+y)^2}$$

$$RHS = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left[1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{2xy + y^2 - x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{(x+y)^2 - x^2 - 2xy + y^2 - 2xy - y^2 + x^2}{(x+y)^2}\right]$$

$$RHS = 4\left[\frac{(x+y)^2 - 4xy}{(x+y)^2}\right] = 4\left[\frac{x^2 + 2xy + y^2 - 4xy}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2 - 2xy + y^2}{(x+y)^2}\right] = 4\left[\frac{(x-y)^2}{(x+y)^2}\right]$$

$$\therefore LHS = RHS. \quad \text{Hence proved.}$$

3. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

<u>Soln</u> :

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \quad —①$$

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3x^2 - 3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3y^2 - 3xz)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3z^2 - 3xy)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 + 3y^2 + 3z^2 - 3xy - 3xz - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - xz - yz)}{x^3 + y^3 + z^3 - 3xyz}$$

$$\left\{x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - xz - yz)\right\}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

$$LHS = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial n}\left(\overline{x+y+z}\right)^{\prime} - \left(\underline{\quad}\right)^{\prime} \quad \underline{\quad}$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$LHS. = \frac{-9}{(x+y+z)^2} = RHS.$$

4. If $\theta = t^n e^{-r^2/4t}$, find n which will make $\frac{\partial\theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)$.

$$\frac{d}{dn}(uv) = u'v + v'u$$

Soln:-
$$\theta = f(t, r)$$
$$\theta = t^n e^{-r^2/4t}$$

Differentiating θ wrt t

$$LHS = \frac{\partial\theta}{\partial t} = \frac{\partial}{\partial t}(t^n)\cdot e^{-r^2/4t} + t^n \cdot \frac{\partial}{\partial t}\left(e^{-r^2/4t}\right)$$

$$= nt^{n-1} e^{-r^2/4t} + t^n \cdot e^{-r^2/4t}\cdot \frac{\partial}{\partial t}\left(-\frac{r^2}{4t}\right)$$

$$= nt^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t}\left(-\frac{r^2}{4}\right)\left(-\frac{1}{t^2}\right)$$

$$LHS = \frac{\partial\theta}{\partial t} = e^{-r^2/4t}\left[nt^{n-1} + \frac{r^2}{4}t^{n-2}\right]$$

Now $\theta = t^n e^{-r^2/4t}$

differentiating wrt r

$$\frac{\partial\theta}{\partial r} = t^n \cdot e^{-r^2/4t}\cdot\frac{\partial}{\partial r}\left(-\frac{r^2}{4t}\right) = t^n e^{-r^2/4t}\left(-\frac{2r}{4t}\right)$$

$$= -\frac{1}{2}t^{n-1}\cdot r\, e^{-r^2/4t}$$

$$r^2\frac{\partial\theta}{\partial r} = -\frac{1}{2}t^{n-1}\, r^3\, e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{1}{2}t^{n-1}\left\{\frac{\partial}{\partial r}(r^3)\cdot e^{-r^2/4t} + r^3\frac{\partial}{\partial r}\left(e^{-r^2/4t}\right)\right\}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) \qquad \qquad$$

$$= -\frac{1}{2} t^{n-1} \left\{ 3r^2 e^{-r^2/4t} + r^3 e^{-r^2/4t} \cdot \frac{\partial}{\partial r}\left(\frac{-r^2}{4t}\right)\right\}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1} \left\{ 3r^2 e^{-r^2/4t} - \frac{r^4}{2t} e^{-r^2/4t}\right\}$$

$$RHS = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1}\left\{ 3 e^{-r^2/4t} - \frac{r^2}{2t} e^{-r^2/4t}\right\}$$

$$RHS. \qquad = e^{-r^2/4t}\left\{ -\frac{3}{2} t^{n-1} + \frac{r^2}{4} t^{h-2}\right\}$$

comparing LHS & RHS.

$$e^{-r^2/4t}\left\{ n t^{n-1} + \frac{r^2}{4} t^{n-2}\right\} = e^{-r^2/4t}\left\{ -\frac{3}{2} t^{n-1} + \frac{r^2}{4} t^{n-2}\right\}$$

By comparing, we get $n = -\frac{3}{2}$.

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5. If $z = x^y + y^x$, verify that $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$.

$\frac{d}{dx}(a^x) = a^x \log a$

$\frac{d}{dx}(y^x) = xy^{x-1}$

Soln:- $\quad z = x^y + y^x$

Differenting wrt y first

$$\frac{\partial z}{\partial y} = x^y \log x + x y^{x-1}$$

differentiating $\frac{\partial z}{\partial y}$ wrt x

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(x^y) \log x + x^y \frac{\partial}{\partial x}(\log x)$$
$$+ \frac{\partial}{\partial x}(x) y^{x-1} + x \frac{\partial}{\partial x}(y^{x-1})$$

$$= y x^{y-1} \log x + x^y \cdot \frac{1}{x} + y^{x-1} + x y^{x-1} \log y$$

$$LHS \quad = y x^{y-1} \log x + x^{y-1} + y^{x-1} + x y^{x-1} \log y$$

$LHS = y^x \log x + x + y + x^y \cdots$

H.W.

$$\frac{\partial z}{\partial x} =$$

$$\frac{\partial^2 z}{\partial y \partial x} =$$

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6. If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ and $a^2 + b^2 + c^2 = 1$, Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Soln:

$u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$

differentiating wrt x twice partially

$$\frac{\partial u}{\partial x} = 6(ax + by + cz) \cdot (a) - 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 6a(a) - 2 = 6a^2 - 2 \quad ——①$$

differentiating wrt y twice

$$\frac{\partial u}{\partial y} = 6(ax + by + cz) \cdot (b) - 2y$$

$$\frac{\partial^2 u}{\partial y^2} = 6(b) \cdot (b) - 2 = 6b^2 - 2 \quad ——②$$

differentiating u wrt z twice

$$\frac{\partial u}{\partial z} = 6(ax + by + cz) \cdot (c) - 2z$$

$$\frac{\partial^2 u}{\partial z^2} = 6c(c) - 2 = 6c^2 - 2 \quad ——③$$

① + ② + ③ =>

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6(a^2 + b^2 + c^2) - 6$$

Now $a^2 + b^2 + c^2 = 1$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6(1) - 6 = 0 \qquad \text{Hence proved}$$

Now

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6(1) - 6 = 0$$ Hence proved.

7. If $u = f(r), r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$.

$\underline{Soln}$:- $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$

differentiating u wrt x

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r} = \frac{f'(r) \cdot x}{r}$$

differentiating wrt x again

$$\frac{\partial^2 u}{\partial x^2} = \frac{r \cdot \frac{\partial}{\partial x}\left(f'(r) \cdot x\right) - f'(r) \cdot x \frac{\partial}{\partial x}(r)}{r^2}$$

$$= \frac{r \cdot \left[f'(r) \cdot \frac{\partial x}{\partial x} + \frac{\partial}{\partial x}\left[f'(r)\right] \cdot x\right] - f'(r) \cdot x \left(\frac{x}{r}\right)}{r^2}$$

$$= \frac{r\left[f'(r)(1) + f''(r) \cdot \frac{\partial r}{\partial x} \cdot x\right] - f'(r) \cdot \frac{x^2}{r}}{r^2}$$

$$= \frac{r\left[f'(r) + f''(r) \cdot \frac{x^2}{r}\right] - f'(r) \cdot \frac{x^2}{r}}{r^2}$$

$$= \frac{r f'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r)}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(r^2 - x^2\right)f'(r) + x^2 r f''(r)}{\quad} \qquad —\!\!—\,①$$

Right side work:

$r^2 = x^2 + y^2 + z^2$
diff wrt x

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{2x}{2r}$$

$$\Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

Similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

Similarly, $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{(r^2 - y^2)\, f'(r) + y^2 r\, f''(r)}{r^3}$ ——②

$\dfrac{\partial^2 u}{\partial z^2} = \dfrac{(r^2 - z^2)\, f'(r) + z^2 r\, f''(r)}{r^3}$ ——③

① + ② + ③

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

$= \dfrac{\left[3r^2 - (x^2 + y^2 + z^2)\right] f'(r) + r\, f''(r)(x^2 + y^2 + z^2)}{r^3}$

$= \dfrac{(3r^2 - r^2)\, f'(r) + r\, f''(r) \cdot r^2}{r^3}$

$= \dfrac{2r^2}{r^3} f'(r) + \dfrac{r^3}{r^3} f''(r)$

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{2}{r} f'(r) + f''(r)$

8. If $u = e^{x^2 + y^2 + z^2}$, prove that $\dfrac{\partial^3 u}{\partial x\, \partial y\, \partial z} = 8\, xyzu.$

Soln :-

$u = e^{x^2 + y^2 + z^2}$

$\dfrac{\partial u}{\partial z} = e^{x^2 + y^2 + z^2} \cdot \dfrac{\partial}{\partial z}(x^2 + y^2 + z^2) = 2z\, e^{x^2 + y^2 + z^2}$

$\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial z}\right) = \dfrac{\partial^2 u}{\partial y\, \partial z} = 2z \cdot \left[e^{x^2 + y^2 + z^2} \cdot \dfrac{\partial}{\partial y}(x^2 + y^2 + z^2)\right]$

$= 2z \left[e^{x^2 + y^2 + z^2} \cdot 2y\right]$

$x^2 u$

$x^2 + y^2 + z^2$

$$\frac{\partial}{\partial y \partial z} = 4yz\, e^{\cdots}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial^3 u}{\partial x \partial y \partial z} = 4yz\left[e^{x^2+y^2+z^2} \cdot \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)\right]$$

$$= 4yz\left[e^{x^2+y^2+z^2} \cdot 2x\right]$$

$$= 8xyz\, e^{x^2+y^2+z^2}$$

$$\underline{\frac{\partial^3 u}{\partial x \partial y \partial z}} = 8\,xyz\, u$$

9. If $z = u(x,y)\, e^{ax+by}$ where $u(x,y)$ is such that $\frac{\partial^2 u}{\partial x \partial y} = 0$, find the constants a, b such

that $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$.

<u>Soln</u> :

$$z = u(x,y)\, e^{ax+by}$$

differentiating wrt x partially

$$\frac{\partial z}{\partial x} = u(x,y)\frac{\partial}{\partial x}\left(e^{ax+by}\right) + \frac{\partial}{\partial x}\left(u(x,y)\right)e^{ax+by}$$

$$= u(x,y)\cdot e^{ax+by}\cdot(a) + \frac{\partial u}{\partial x}e^{ax+by}$$

$$\frac{\partial z}{\partial x} = e^{ax+by}\left(au + \frac{\partial u}{\partial x}\right) \quad\text{——}\enspace①$$

differentiating z wrt y

$$z = u(x,y)\, e^{ax+by}$$

$$\frac{\partial z}{\partial y} = u(x,y)\frac{\partial}{\partial y}\left(e^{ax+by}\right) + \frac{\partial}{\partial y}\left(u(x,y)\right)\cdot e^{ax+by}$$

$$= u(x,y)\cdot e^{ax+by}\cdot(b) + \frac{\partial u}{\partial y}e^{ax+by}$$

$$\frac{\partial z}{\partial y} = e^{ax+by}\left[bu + \frac{\partial u}{\partial y}\right] \qquad \text{——} \quad ②$$

differentiating $\frac{\partial z}{\partial y}$ partially wrt x

$$\frac{\partial^2 z}{\partial x \partial y} = e^{ax+by} \cdot \frac{\partial}{\partial x}\left[bu + \frac{\partial u}{\partial y}\right] + \left[bu + \frac{\partial u}{\partial y}\right] \cdot \frac{\partial}{\partial x}\left(e^{ax+by}\right)$$

$$= e^{ax+by} \cdot \left[b \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y}\right] + \left[bu + \frac{\partial u}{\partial y}\right] \cdot e^{ax+by} \cdot (a)$$

$$= e^{ax+by}\left[abu + a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y}\right]$$

$$\text{but } \frac{\partial^2 u}{\partial x \partial y} = 0$$

$$\therefore \quad \frac{\partial^2 z}{\partial x \partial y} = e^{ax+by}\left[abu + a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x}\right] \quad \text{——} \quad ③$$

Now we are given that

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$$

Using ① , ② , ③

$$e^{ax+by}\left[abu + a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x}\right] - e^{ax+by}\left[au + \frac{\partial u}{\partial x}\right]$$

$$- e^{ax+by}\left[bu + \frac{\partial u}{\partial y}\right] + e^{ax+by} \cdot u = 0$$

$$e^{ax+by}\left[\underline{abu} + a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x} - \underline{au} - \frac{\partial u}{\partial x} - \underline{bu} - \frac{\partial u}{\partial y} + u\right] = 0$$

$$e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + (a-1)(b-1)u\right] = 0$$

$$e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + (a-1)(b-1)u\right] = 0$$

$$e^{ax+by} \neq 0, \quad \frac{\partial u}{\partial x} \neq 0, \quad \frac{\partial u}{\partial y} \neq 0, \quad u = 0$$

$$\Rightarrow a-1 = 0, \quad b-1 = 0 \quad \Rightarrow a=1, b=1 .$$

10. If $a^2x^2 + b^2y^2 = c^2z^2$, evaluate $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$

Soln:- $\quad c^2z^2 = a^2x^2 + b^2y^2 \quad \Rightarrow \quad z^2 = \frac{a^2}{c^2}x^2 + \frac{b^2}{c^2}y^2$

differentiating wrt x

$$2z \cdot \frac{\partial z}{\partial x} = \frac{a^2}{c^2}(2x) + 0$$

$$\frac{\partial z}{\partial x} = \frac{a^2}{c^2} \cdot \frac{x}{z}$$

Differentiating $\frac{\partial z}{\partial x}$ wrt x partially

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2} \left[\frac{z \cdot \frac{\partial}{\partial x}(x) - x \cdot \frac{\partial}{\partial x}(z)}{z^2} \right]$$

$$= \frac{a^2}{c^2} \cdot \frac{z \cdot (1) - x \cdot \frac{\partial z}{\partial x}}{z^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2}\left[\frac{z - x \cdot \frac{a^2}{c^2}\left(\frac{x}{z}\right)}{z^2} \right]$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2}\left[\frac{c^2z^2 - a^2x^2}{c^2 z^3} \right]$$

$$\frac{1}{a^2} \frac{\partial^2 z}{\partial x^2} = \frac{1}{c^4}\left[\frac{c^2 z^2 - a^2 x^2}{z^3}\right] \quad - \textcircled{1}$$

Similarly,

$$\frac{1}{b^2} \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^4}\left[\frac{c^2 z^2 - b^2 y^2}{z^3}\right] \quad - \textcircled{2}$$

$\textcircled{1} + \textcircled{2}$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^4}\left[\frac{2c^2 z^2 - (a^2 x^2 + b^2 y^2)}{z^3}\right]$$

$$= \frac{1}{c^4}\left[\frac{2c^2 z^2 - c^2 z^2}{z^3}\right]$$

$$= \frac{1}{c^4}\left[\frac{c^2 z^2}{z^3}\right]$$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2 z}$$