CONCEPT- PARTIAL DIFFERENTIATION

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f_x

Partial Derivatives of the first order:

Let $z = f(x, y)$ be a function of two independent variables x and y.

If we keep y constant and allow only x to vary then derivative, if it exists so obtained is called the **partial derivative of z with respect to x** and it is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x.

thus, $\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$

Similarly, the derivative of z with respect to y keeping x constant, if it exists is called the **partial derivative of z with respect to y** and it is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y.

thus, $\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$

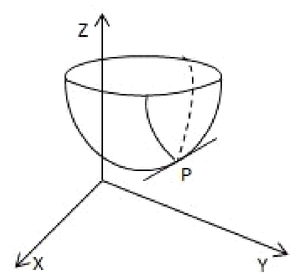
Geometrical Interpretation of Partial Derivatives:

We know that $z = f(x, y)$ represents a surface in space. If $x = k, z = f(k, y)$ represents a curve of intersection of $z = f(x, y)$ and the plane $x = k$.

Similarly if $y = k, z = f(x, k)$ represents a curve of intersection of $z = f(x, y)$and the plane $y = k$

$\therefore \frac{\partial z}{\partial x}$ represents the slope of the tangent to the curve of intersection of the surface $z = f(x, y)$and a plane $y = k.$ (see the figure)

Similarly $\frac{\partial z}{\partial y}$ represents the slope of the tangent to the curve of interaction of the surface $z = f(x, y)$and a plane $x = k.$



Partial Derivatives of Higher Order:

The partial derivatives of higher order, if they exist, can be obtained from partial derivatives of the first order by using the above definitions again.

Thus, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$ is the second order partial derivative of z w.r.t. x and is denoted by $\frac{\partial^2 z}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}.

Similarly, we have $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x} = f_{yx}$$

And $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$

Note: (1) If $u = f(x, y)$ possesses continuous second-order partial derivatives $\frac{\partial^2 x}{\partial x \partial y}$ and $\frac{\partial^2 x}{\partial y \partial x}$ then $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

This is called commutative property

(2) Standard rules for differentiation of sum, difference, product and quotient are also applicable for partial differentiation

Differentiation of a Function of a function:

Let $z = f(u)$and $u = \Phi(x, y)$ so that z is function of u and u itself is a function of two independent variables x and y. The two relations define z as a function of x and y. In such cases z may be called a **function of a function** of x and y.

e.g. **(i)** $z = \frac{1}{u}$ and $u = \sqrt{x^2 + y^2}$ **(ii)** $z = \tan u$ and $u = x^2 + y^2$

define z as a function of a function of x and y.

Differentiation: If $z = f(u)$ is differentiable function of u and $u = \Phi(x, y)$possesses first order partial derivatives then, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{dx}$ i.e. $\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x}$

Similarly $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y}$ i.e. $\frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$

e.g. If $z = (ax + by)^n$ then $\frac{\partial z}{\partial x} = n(ax + by)^{n-1}. a$ and $\frac{\partial z}{\partial y} = n(ax + by)^{n-1}. b$

① $z = \frac{1}{(x+y)^2} \rightarrow (x+y)^{-2}$ $\quad x^n \quad nx^{n-1}$

$$\frac{\partial z}{\partial x} = \frac{-2}{(x+y)^3}(1) \qquad \frac{\partial z}{\partial y} = \frac{-2}{(x+y)^3} \cdot (1)$$

② $z = \tan(x^2y^2)$

$$\frac{\partial z}{\partial x} = \sec^2(x^2y^2) \cdot (2xy^2) \qquad \frac{\partial z}{\partial y} = \sec^2(x^2y^2)(x^2 \cdot 2y)$$

③ $z = \log(x^2y^2 + 3xy)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2y^2 + 3xy} \cdot (2xy^2 + 3y) \qquad \frac{\partial z}{\partial y} = \frac{1}{x^2y^2 + 3xy} \cdot (2yx^2 + 3x)$$

④ $u = e^{x^2+y^2+z^2}$

$$\frac{\partial u}{\partial x} = e^{x^2+y^2+z^2} \cdot (2x + 0 + 0)$$

$$\frac{\partial u}{\partial y} = e^{x^2+y^2+z^2}(0 + 2y + 0)$$

$$\frac{\partial u}{\partial z} = e^{x^2+y^2+z^2}(0 + 0 + 2z)$$

(5) $z = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot (2x)$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot (2y)$$

$$= \frac{y}{\sqrt{x^2+y^2}}$$