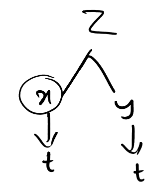
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \times \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt}$$



COMPOSITE FUNCTIONS

(a) Let $z = f(x,y)$ and $x = \Phi(t)$, $y = \Psi(t)$ so that z is function of x ,y and x ,y are function of third variable t. The three relations define z as a function of t. In such cases z is called a **composite function of t**.

e.g. **(i)** $z = x^2 + y^2$, $x = at^2$, $y = 2at$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x)(2at) + (2y)(2a)$$

$$= (2at^2)(2at) + (4at)(2a)$$

$$= 4a^2t^3 + 8a^2t$$

$$z = x^2 + y^2$$

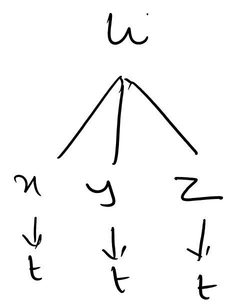
$$= (at^2)^2 + (2at)^2$$

$$z = a^2t^4 + 4a^2t^2$$

$$\frac{dz}{dt} = 4a^2t^3 + 8a^2t$$

(ii) $z = x^2y + xy^2$, $x = a\cos t$, $y = b\sin t$ define z as a composite function of t

Differentiation: Let $z = f(x,y)$ possesses continuous first order partial derivatives and $x = \Phi(t), y = \Psi(t)$ possesses continuous first order derivatives then, $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$



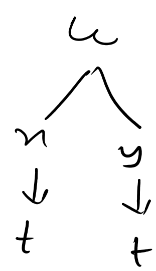
$u = f(x, y, z)$ $x \to \phi_1(t)$, $y = \phi_2(t)$, $z = \phi_3(t)$

$u \to$ composite f^n of t

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

SOME SOLVED EXAMPLES:

1. If $u = x^2y^3, x = \log t, y = e^t$, find $\frac{du}{dt}$



Soln.:-

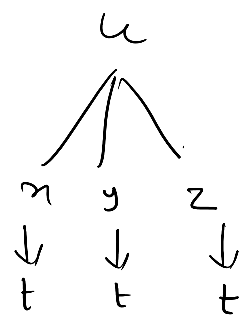
$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2xy^3)\left(\frac{1}{t}\right) + (3x^2y^2)(e^t)$$

$$= (2 \log t \, e^{3t})\left(\frac{1}{t}\right) + (3(\log t)^2 e^{2t}) e^t$$

$$\frac{du}{dt} = e^{3t}\left[\frac{2\log t}{t} + 3(\log t)^2\right]$$

2. If $u = xy + yz + zx$ where $x = \frac{1}{t}, y = e^t, z = e^{-t}$, find $\frac{du}{dt}$



$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

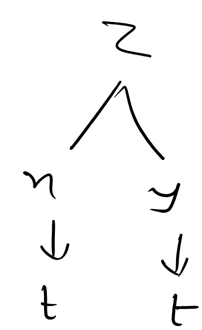
$$= (y+z)\left(\frac{-1}{t^2}\right) + (x+z)(e^t) + (x+y)(-e^{-t})$$

$$= (e^t + e^{-t})\left(\frac{-1}{t^2}\right) + \left(\frac{1}{t} + e^{-t}\right)e^t + \left(\frac{1}{t} + e^t\right)(-e^{-t})$$

$$= (e^t + e^{-t})\left(\frac{-1}{t^2}\right) + \frac{e^t}{t} + 1 - \frac{e^{-t}}{t} - 1$$

$$\frac{du}{dt} = \frac{1}{t}(e^t - e^{-t}) - \frac{1}{t^2}(e^t + e^{-t})$$

3. If $z = e^{xy}, x = t\cos t, y = t\sin t$, find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= y e^{xy}\cdot(\cos t - t\sin t)$$

$$+ x e^{xy}(\sin t + t\cos t)$$

at $t = \frac{\pi}{2}$, $x = 0$, $y = \frac{\pi}{2}$

$$\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = \frac{\pi}{2}e^0\left(0 - \frac{\pi}{2}\right) + 0$$

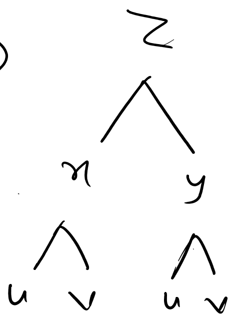
$$= \frac{\pi}{2}\left(-\frac{\pi}{2}\right) = -\frac{\pi^2}{4}$$

(b) Let $z = f(x, y)$ and $x = \Phi(u, v)$, $y = \Psi(u, v)$ so that z is function of x, y and x, y are function of u, v. The three relations define z as a function of u, v. In such cases z is called a **composite function of u, v.**

e.g. **(i)** $z = xy$, $x = e^{u} + e^{-v}$, $y = e^{-u} + e^{v}$

(ii) $z = x^2 - y^2$, $x = 2u - 3v$, $y = 3u + 2v$ define z as a composite function of u and v

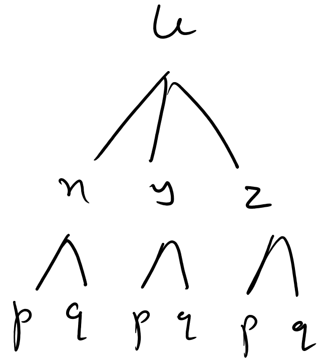
1



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

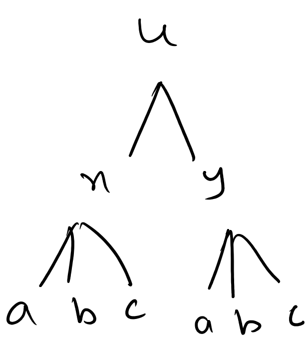
2



$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial p} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial p} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial p}$$

$$\frac{\partial u}{\partial q} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial q} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial q}$$

3

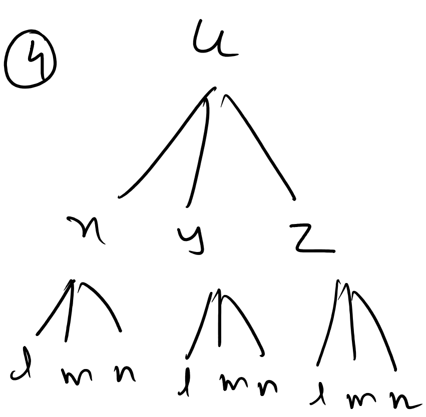


$$\frac{\partial u}{\partial a} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial a} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial a}$$

$$\frac{\partial u}{\partial b} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial b} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial b}$$

$$\frac{\partial u}{\partial c} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial c} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial c}$$

4



$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial l} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial l} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial l}$$

$$\frac{\partial u}{\partial m} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial m} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial m} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial m}$$

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial n} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial n} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial n}$$

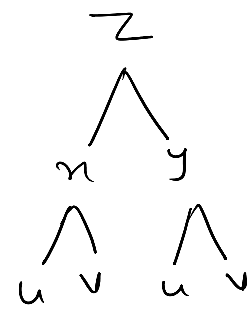
Differentiation: Let $z = f(x, y)$ possesses continuous first order partial derivatives and $x = \Phi(u, v)$, $y = \Psi(u, v)$ possesses continuous first order partial derivatives then,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

1. If $x^2 = au + bv$, $y^2 = au - bv$ and $z = f(x, y)$, Prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$.

Soln:-

$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \end{aligned}\right\} \text{— (1)}$$



$x^2 = au + bv$

$$\left.\begin{aligned} 2x\frac{\partial x}{\partial u} &= a \Rightarrow \frac{\partial x}{\partial u} = \frac{a}{2x} \\ 2x\frac{\partial x}{\partial v} &= b \Rightarrow \frac{\partial x}{\partial v} = \frac{b}{2x} \end{aligned}\right\} \text{— (2)}$$

$y^2 = au - bv$

$$\left.\begin{aligned} 2y\frac{\partial y}{\partial u} &= a \Rightarrow \frac{\partial y}{\partial u} = \frac{a}{2y} \\ 2y\frac{\partial y}{\partial v} &= -b \Rightarrow \frac{\partial y}{\partial v} = \frac{-b}{2y} \end{aligned}\right\} \text{(3)}$$

Using (1), (2), (3)

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{a}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{a}{2y}\right)$$

$$\therefore u\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{au}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{au}{2y}\right) = \left(\frac{au}{2}\right)\left[\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y}\right]$$

also $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\left(\frac{b}{2x}\right) + \frac{\partial z}{\partial y}\cdot\left(\frac{-b}{2y}\right)$$

$$v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(\frac{bv}{2x}\right) + \frac{\partial z}{\partial y}\left(\frac{-bv}{2y}\right) = \left(\frac{bv}{2}\right)\left[\frac{1}{x}\frac{\partial z}{\partial x} - \frac{1}{y}\frac{\partial z}{\partial y}\right]$$

$$\therefore RHS = 2\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$$

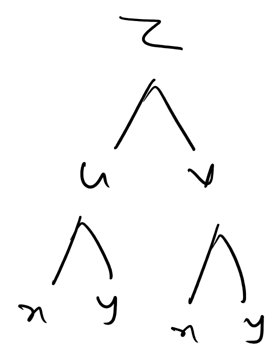
$$= au\left[\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y}\right] + bv\left[\frac{1}{x}\frac{\partial z}{\partial x} - \frac{1}{y}\frac{\partial z}{\partial y}\right]$$

$$= \frac{1}{x}\frac{\partial z}{\partial x}(au+bv) + \frac{1}{y}\frac{\partial z}{\partial y}(au-bv)$$

$$= \frac{1}{x}\cdot\frac{\partial z}{\partial x}(x^2) + \frac{1}{y}\frac{\partial z}{\partial y}(y^2)$$

$$= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = LHS.$$ Hence proved.

2. If $u = \log(x^2+y^2)$, $v = \frac{y}{x}$, prove that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1+v^2)\frac{\partial z}{\partial v}$



Soln :-

$$\left.\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\end{aligned}\right\} \text{— (1)}$$

$$u = \log(x^2+y^2)$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{2x}{x^2+y^2}\\ \frac{\partial u}{\partial y} &= \frac{2y}{x^2+y^2}\end{aligned}\right\} \text{— (2)}$$

$$v = \frac{y}{x}$$

$$\left.\begin{aligned}\frac{\partial v}{\partial x} &= -\frac{y}{x^2}\\ \frac{\partial v}{\partial y} &= \frac{1}{x}\end{aligned}\right\} \text{— (3)}$$

using (1), (2), (3)

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\left(\frac{2x}{x^2+y^2}\right) + \frac{\partial z}{\partial v}\cdot\left(\frac{-y}{x^2}\right)$$

$$\therefore y\frac{\partial z}{\partial x} = \frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} - \frac{y^2}{x^2}\frac{\partial z}{\partial v} \quad \text{—— (4)}$$

also $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\left(\frac{2y}{x^2+y^2}\right) + \frac{\partial z}{\partial v}\cdot\left(\frac{1}{x}\right)$$

— (5)

also $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \left(x^2+y^2 \right) \ldots$

$$\therefore x \frac{\partial z}{\partial y} = \frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{—— (5)}$$

(5) − (4) will give

$$LHS = x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}$$

$$= \left[\frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right] - \left[\frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} - \frac{y^2}{x^2}\frac{\partial z}{\partial v}\right]$$

$$= \left(1+\frac{y^2}{x^2}\right)\frac{\partial z}{\partial v} = \left(1+v^2\right)\frac{\partial z}{\partial v} = RHS.$$

Hence proved.

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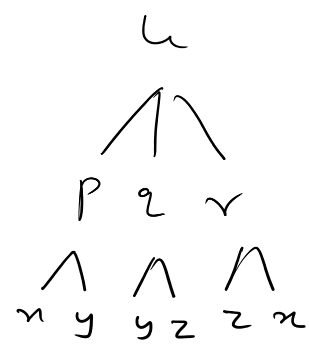
3. If $u = f\left(x^2-y^2, y^2-z^2, z^2-x^2\right)$, prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$.

Soln :- let $p = x^2-y^2$, $q = y^2-z^2$, $r = z^2-x^2$

$u = f(p, q, r)$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}\cdot(2x) + \frac{\partial u}{\partial r}\cdot(-2x)$$

$$\therefore \frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-2y) + \frac{\partial u}{\partial q}\cdot(2y)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-2y) + \frac{\partial u}{\partial q}\cdot(2y)$$

$$\frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial p} + 2\frac{\partial u}{\partial q} \qquad \text{——} \quad (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial q}(-2z) + \frac{\partial u}{\partial r}(2z)$$

$$\frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial q} + 2\frac{\partial u}{\partial r} \qquad \text{——} \quad (3)$$

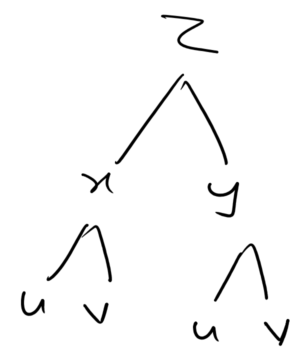
adding (1), (2) & (3)

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial p} + 2\frac{\partial u}{\partial q}$$

$$- 2\frac{\partial u}{\partial q} + 2\frac{\partial u}{\partial r} = 0 = \text{RHS.}$$

4. If $x = e^u cosec\, v, y = e^u \cot v\ and\ z$ is a function of x and y, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$$



Sol^n :- $x = e^u \operatorname{cosec} v, \quad y = e^u \cot v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(e^u \operatorname{cosec} v\right) + \frac{\partial z}{\partial y}\cdot\left(e^u \cot v\right)$$

$$\left(\frac{\partial z}{\partial u}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\left(e^{2u}\operatorname{cosec}^2 v\right) + \left(\frac{\partial z}{\partial y}\right)^2\left(e^{2u}\cot^2 v\right)$$

$$+2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(e^{2u}\operatorname{cosec} v \cot v\right) \quad \text{—(1)}$$

Now $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$

$$= \frac{\partial z}{\partial x}\cdot\left(-e^{u}\operatorname{cosec} v \cot v\right) + \frac{\partial z}{\partial y}\left(-e^{u}\operatorname{cosec}^2 v\right)$$

$$\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\left(e^{2u}\operatorname{cosec}^2 v \cot^2 v\right) + \left(\frac{\partial z}{\partial y}\right)^2\left(e^{2u}\operatorname{cosec}^4 v\right)$$

$$+2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(e^{2u}\operatorname{cosec}^3 v \cot v\right)$$

$$(\sin^2 v)\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\left(e^{2u}\cot^2 v\right) + \left(\frac{\partial z}{\partial y}\right)^2\left(e^{2u}\operatorname{cosec}^2 v\right)$$

$$+2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(e^{2u}\operatorname{cosec} v \cot v\right) \quad \text{—(2)}$$

(1) − (2)

$$\left(\frac{\partial z}{\partial u}\right)^2 - (\sin^2 v)\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\left(\operatorname{cosec}^2 v - \cot^2 v\right)$$

$$+\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\left(\cot^2 v - \operatorname{cosec}^2 v\right)$$

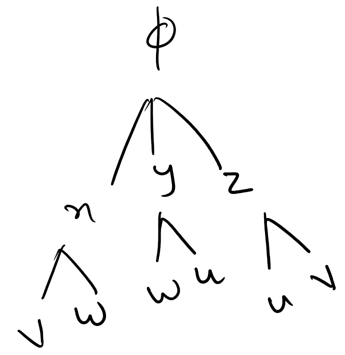
$$\therefore \text{RHS} = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - (\sin^2 v)\left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\left(\operatorname{cosec}^2 v - \cot^2 v\right) + \left(\frac{\partial z}{\partial y}\right)^2\left(\cot^2 v - \operatorname{cosec}^2 v\right)$$

$$\text{RHS} = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \text{LHS}$$

5 If $x = \sqrt{vw}, y = \sqrt{wu}, z = \sqrt{uv}$, prove that

$$x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y} + z\frac{\partial \Phi}{\partial z} = u\frac{\partial \Phi}{\partial u} + v\frac{\partial \Phi}{\partial v} + w\frac{\partial \Phi}{\partial w}$$ where Φ is a function of x, y, z.



Soln :-

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \frac{\partial \phi}{\partial y}\cdot\frac{\sqrt{w}}{2\sqrt{u}} + \frac{\partial \phi}{\partial z}\cdot\frac{\sqrt{v}}{2\sqrt{u}}$$

$y = \sqrt{wu}$

$\frac{\partial y}{\partial w} = \sqrt{w}\left(\frac{1}{2\sqrt{u}}\right)$

$= \frac{\sqrt{w}}{2\sqrt{u}}$

$$\therefore u\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial y}\cdot\frac{\sqrt{uw}}{2} + \frac{\partial \phi}{\partial z}\cdot\frac{\sqrt{uv}}{2}$$

$$u\frac{\partial \phi}{\partial u} = \frac{y}{2}\frac{\partial \phi}{\partial y} + \frac{z}{2}\frac{\partial \phi}{\partial z} \quad \text{— (1)}$$

$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= \frac{\partial \phi}{\partial x}\cdot\frac{\sqrt{w}}{2\sqrt{v}} + \frac{\partial \phi}{\partial z}\cdot\frac{\sqrt{u}}{2\sqrt{v}}$$

$$v\frac{\partial \phi}{\partial v} = \frac{\sqrt{vw}}{2}\frac{\partial \phi}{\partial x} + \frac{\sqrt{uv}}{2}\frac{\partial \phi}{\partial z} = \frac{x}{2}\frac{\partial \phi}{\partial x} + \frac{z}{2}\frac{\partial \phi}{\partial z} \quad \text{— (2)}$$

$$\frac{\partial \phi}{\partial w} = \frac{\partial \phi}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial \phi}{\partial y} + \frac{\partial y}{\partial w}$$

$$= \frac{\partial \phi}{\partial x}\cdot\frac{\sqrt{v}}{2\sqrt{w}} + \frac{\partial \phi}{\partial y}\cdot\frac{\sqrt{u}}{2\sqrt{w}}$$

$$w\frac{\partial \phi}{\partial w} = \frac{\sqrt{vw}}{2}\frac{\partial \phi}{\partial x} + \frac{\sqrt{uw}}{2}\frac{\partial \phi}{\partial y} = \frac{x}{2}\frac{\partial \phi}{\partial x} + \frac{y}{2}\frac{\partial \phi}{\partial y} \quad \text{— (3)}$$

adding (1), (2) & (3)

adding ① , ② & ③

$$u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v} + w\frac{\partial\phi}{\partial w} = x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + z\frac{\partial\phi}{\partial z}.$$