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Definition: If u and v are functions of two independent variables x and y then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ or } \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

is called the Jacobian of u and v with respect to x and y and it is denoted as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right) \rightarrow J$

Similarly, If $u, v, w \rightarrow x, y, z$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Property: If J denotes the Jacobian of u, v with respect to x, y and J' denotes the Jacobian of x, y with respect to u, v then it can be proved that $JJ' = 1$

SOME SOLVED EXAMPLES:

1. Prove that $JJ' = 1$ for $x = e^v \sec u, y = e^v \tan u$

Soln :- $x = e^v \sec u, \quad y = e^v \tan u$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} e^v \sec u \tan u & e^v \sec u \end{vmatrix}$$

$$= \begin{vmatrix} e^{v}\sec u\tan u & e^{v}\sec u \\ e^{v}\sec^2 u & e^{v}\tan u \end{vmatrix}$$

$$= e^{2v}\sec u\tan^2 u - e^{2v}\sec^3 u$$

$$= e^{2v}\sec u(\tan^2 u - \sec^2 u)$$

$$J = -e^{2v}\sec u = -(e^{v}\sec u)e^{v} = -x e^{v} \quad (1)$$

Now, $J' = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

$x = e^{v}\sec u, \quad y = e^{v}\tan u$

$$\frac{y}{x} = \frac{e^{v}\tan u}{e^{v}\sec u} = \sin u \Rightarrow \boxed{u = \sin^{-1}\left(\frac{y}{x}\right)}$$

$$x^2 - y^2 = e^{2v}(\sec^2 u - \tan^2 u) = e^{2v}$$

$$\Rightarrow 2v = \log(x^2 - y^2) \Rightarrow \boxed{v = \frac{1}{2}\log(x^2 - y^2)}$$

Now $\dfrac{\partial u}{\partial x} = \dfrac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \left(\dfrac{-y}{x^2}\right) = \dfrac{-y}{x\sqrt{x^2-y^2}}$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \left(\frac{1}{x}\right) = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial v}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2-y^2} \cdot 2x = \frac{x}{x^2-y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2-y^2} \cdot (-2y) = \frac{-y}{x^2-y^2}$$

$$\frac{\partial y}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2-y^2} \cdot (-2y) = \frac{-y}{x^2-y^2}$$

$$\therefore J' = \begin{vmatrix} \dfrac{-y}{x\sqrt{x^2-y^2}} & \dfrac{1}{\sqrt{x^2-y^2}} \\ \dfrac{x}{x^2-y^2} & \dfrac{-y}{x^2-y^2} \end{vmatrix}$$

$$= \frac{y^2}{x(x^2-y^2)^{3/2}} - \frac{x}{(x^2-y^2)^{3/2}}$$

$$= \frac{(y^2-x^2)}{x(x^2-y^2)^{3/2}} = \frac{-1}{x\sqrt{x^2-y^2}}$$

but $x^2-y^2 = e^{2v}$

$$\therefore J' = \frac{-1}{x e^v} \quad \text{———} \quad (2)$$

$$\therefore JJ' = (-x e^v)\left(\frac{-1}{x e^v}\right) = 1.$$

2. Prove that $JJ' = 1$ for $x = \sin\theta\cos\Phi$, $y = \sin\theta\sin\Phi$

Soln:- $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \end{vmatrix}$$

$$= \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi$$

$$J = \sin\theta\cos\theta(\cos^2\phi+\sin^2\phi) = \sin\theta\cos\theta \longrightarrow (1)$$

Now $\dfrac{y}{x} = \dfrac{\sin\theta\sin\phi}{\sin\theta\cos\phi} = \tan\phi \Rightarrow \phi = \tan^{-1}\left(\dfrac{y}{x}\right)$

$$x^2+y^2 = \sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\theta = \sin^2\theta$$

$$\Rightarrow \theta = \sin^{-1}\sqrt{x^2+y^2}$$

$$\left(\frac{d}{dt}\sin^{-1}t\right) = \frac{1}{\sqrt{1-t^2}}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{\sqrt{1-(x^2+y^2)}}\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot(2x)$$

$$\frac{\partial\theta}{\partial x} = \frac{x}{\sqrt{1-x^2-y^2}\,\sqrt{x^2+y^2}}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{\sqrt{1-(x^2+y^2)}}\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot(2y) = \frac{y}{\sqrt{1-x^2-y^2}\,\sqrt{x^2+y^2}}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\therefore J' = \begin{vmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{1-x^2-y^2}\,\sqrt{x^2+y^2}} & \frac{y}{\sqrt{1-x^2-y^2}\,\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{\sqrt{1-x^2-y^2}\,(x^2+y^2)^{3/2}} + \frac{y^2}{\sqrt{1-x^2-y^2}\,(x^2+y^2)^{3/2}}$$

$$= \frac{(x^2+y^2)}{\sqrt{1-(x^2+y^2)}\,(x^2+y^2)^{3/2}}$$

$$= \frac{1}{\sqrt{1-(x^2+y^2)}\,\sqrt{x^2+y^2}}$$

$$J' = \frac{1}{\sqrt{1-\sin^2\theta}\,\sqrt{\sin^2\theta}} = \frac{1}{\sin\theta\cos\theta} \quad \text{--- (2)}$$

$$\therefore JJ' = (\sin\theta\cos\theta)\frac{1}{\sin\theta\cos\theta} = 1.$$

3. Prove that $JJ' = 1$ for $x = \sqrt{vw}, y = \sqrt{wu}, z = \sqrt{uv}$

Sol^n. $$J = \frac{\partial(x\ y\ z)}{\partial(u, v\ w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \frac{\sqrt{w}}{2\sqrt{v}} & \frac{\sqrt{v}}{2\sqrt{w}} \\ \frac{\sqrt{w}}{2\sqrt{u}} & 0 & \frac{\sqrt{u}}{2\sqrt{w}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} & 0 \end{vmatrix}$$

$$= 0 - \frac{\sqrt{w}}{\ldots}\left[0 - \frac{\sqrt{v}}{\ldots}\right] + \frac{\sqrt{v}}{\ldots}\left[\frac{\sqrt{w}}{\ldots} - 0\right]$$

$$= 0 - \frac{\sqrt{w}}{2\sqrt{v}}\left[0 - \frac{\sqrt{v}}{4\sqrt{w}}\right] + \frac{\sqrt{v}}{2\sqrt{w}}\left[\frac{\sqrt{w}}{4\sqrt{v}} - 0\right]$$

$$J = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$

$$\frac{xz}{y} = \frac{\sqrt{uv^2w}}{\sqrt{uw}} = v \qquad \boxed{v = \frac{xz}{y}}$$

$$\frac{xy}{z} = \frac{\sqrt{uvw^2}}{\sqrt{uv}} = w \qquad \boxed{w = \frac{xy}{z}}$$

$$\frac{yz}{x} = \frac{\sqrt{u^2vw}}{\sqrt{vw}} = u \qquad \boxed{u = \frac{yz}{x}}$$

$$J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$J' = 4$$

$$\therefore \; J J' = \frac{1}{4} \times 4 = 1$$

4. If $u = \frac{y^2}{2x}, v = \frac{x^2+y^2}{2x}$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol^n :-

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{-y^2}{2x^2} & \frac{y}{x} \\ \frac{x^2 - y^2}{2x^2} & \frac{y}{x} \end{vmatrix}$$

$=$

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$$= \frac{-y}{2x}$$

HW **~~5.~~** If $x = a\cos hu \cos v$, $y = a\sin hu \, \sin v$, show that $\frac{\partial(x,y)}{\partial(u,v)} = \frac{a^2}{2}[\cos h2u - \cos 2v]$

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6. If $u = xyz, v = x^2+y^2+z^2, w = x+y+z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)} = J$

We will find
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(2y-2z) - xz(2x-2z) + xy(2x-2y)$$

$$= 2y^2z - \underline{2yz^2} - 2x^2z + \underline{2xz^2} + \underline{2x^2y} - \underline{2xy^2}$$

$$= 2\left[(xz^2 - yz^2) + y^2z - zx^2 + x^2y - xy^2\right]$$

$$= 2\left[z^2(x-y) - z(x^2-y^2) + xy(x-y)\right]$$

$$= 2\left[z^2(x-y) - z(x+y)(x-y) + xy(x-y)\right]$$

$$= 2(x-y)\left[z^2 - zx - zy + xy\right]$$

$$= 2(x-y)\left[z(z-x) - y(z-x)\right]$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 2(x-y)(z-x)(z-y)$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{1}{2(x-y)(z-x)(z-y)}$$

7. If $ux = yz, vy = zx, wz = xy$, prove that $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ is constant

Soln:- $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left[\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-xyz}{yz^2} - \frac{xy}{yz}\right] + \frac{y}{x}\left[\frac{xz}{yz} + \frac{xyz}{y^2z}\right]$$

$$= \frac{-yz}{x^2}\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-x}{z} - \frac{x}{z}\right] + \frac{y}{x}\left[\frac{x}{y} + \frac{x}{y}\right]$$

$$= 0 + 1 + 1 + 1 + 1$$

$$= 4$$

$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)}$ is a constant.