EXAMPLES

Friday, January 7, 2022 2:12 PM

1. If $u = \cos(\sqrt{x} + \sqrt{y})$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}(\sqrt{x} + \sqrt{y})sin(\sqrt{x} + \sqrt{y}) = 0.$

Soln :- $u = \cos(\sqrt{x} + \sqrt{y})$

differentiate u partially wrt x

$$\frac{\partial u}{\partial x} = -\sin(\sqrt{x} + \sqrt{y}) \cdot \frac{\partial}{\partial x}(\sqrt{x} + \sqrt{y})$$

$$= -\sin(\sqrt{x} + \sqrt{y})\left(\frac{1}{2\sqrt{x}} + 0\right) = \frac{-1}{2\sqrt{x}}\sin(\sqrt{x} + \sqrt{y})$$

differentiate u partially wrt y

$$\frac{\partial u}{\partial y} = -\sin(\sqrt{x} + \sqrt{y}) \cdot \frac{\partial}{\partial y}(\sqrt{x} + \sqrt{y})$$

$$= -\sin(\sqrt{x} + \sqrt{y})\left(0 + \frac{1}{2\sqrt{y}}\right) = \frac{-1}{2\sqrt{y}}\sin(\sqrt{x} + \sqrt{y})$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left(\frac{-1}{2\sqrt{x}}\sin(\sqrt{x} + \sqrt{y})\right) + y\left(\frac{-1}{2\sqrt{y}}\sin(\sqrt{x} + \sqrt{y})\right)$$

$$= -\frac{\sqrt{x}}{2}\sin(\sqrt{x} + \sqrt{y}) - \frac{\sqrt{y}}{2}\sin(\sqrt{x} + \sqrt{y})$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\sin(\sqrt{x} + \sqrt{y})\left[\sqrt{x} + \sqrt{y}\right]$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}(\sqrt{x} + \sqrt{y})\sin(\sqrt{x} + \sqrt{y}) = 0.$$

Hence proved.

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

2. If $z(x + y) = x^2 + y^2$, prove that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

Soln :- $z = \dfrac{x^2 + y^2}{x + y}$

differentiating z, partially wrt x

$$\frac{\partial z}{\partial x} = \frac{(x+y) \cdot \frac{\partial}{\partial x}(x^2 + y^2) - (x^2 + y^2) \cdot \frac{\partial}{\partial x}(x + y)}{(x + y)^2}$$

$$= \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

differentiating z partially wrt y

$$\frac{\partial z}{\partial y} = \frac{(x+y)\frac{\partial}{\partial y}(x^2+y^2) - (x^2+y^2)\cdot\frac{\partial}{\partial y}(x+y)}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)(2y) - (x^2+y^2)(1)}{(x+y)^2} = \frac{2xy+y^2-x^2}{(x+y)^2}$$

$$LHS = \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{2xy+y^2-x^2}{(x+y)^2}\right]^2$$

$$= \left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2 = \left[\frac{2(x+y)(x-y)}{(x+y)^2}\right]^2$$

$$\therefore LHS = 4\frac{(x-y)^2}{(x+y)^2}$$

$$RHS = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{2xy+y^2-x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{(x+y)^2 - \cancel{x^2} - 2xy + \cancel{y^2} - 2xy - \cancel{y^2} + \cancel{x^2}}{(x+y)^2}\right]$$

$$RHS = 4\left[\frac{(x+y)^2 - 4xy}{(x+y)^2}\right] = 4\left[\frac{x^2+2xy+y^2-4xy}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right] = 4\left[\frac{(x-y)^2}{(x+y)^2}\right]$$

$\therefore$ LHS = RHS. Hence proved.

3. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Soln:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \quad \text{--- (1)}$$

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3x^2 - 3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3y^2 - 3xz)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3z^2 - 3xy)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 + 3y^2 + 3z^2 - 3xy - 3xz - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - xz - yz)}{x^3 + y^3 + z^3 - 3xyz}$$

$$\left\{x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - xz - yz)\right\}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

$$\text{LHS} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{\ldots}\right) + \frac{\partial}{\partial y}\left(\frac{3}{\ldots}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$-\frac{\partial}{\partial x}\left(\frac{\quad}{x+y+z}\right) \quad - \quad (\qquad \qquad \qquad ($$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$\text{LHS.} = \frac{-9}{(x+y+z)^2} = \text{RHS.}$$

4. If $\theta = t^n e^{-r^2/4t}$, find n which will make $\frac{\partial\theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\theta}{\partial r}\right)$.

$$\frac{d}{dx}(uv) = u'v + v'u$$

Soln:-

$$\theta = f(t, r)$$

$$\theta = t^n e^{-r^2/4t}$$

Differentiating θ wrt t

$$\text{LHS} = \frac{\partial\theta}{\partial t} = \frac{\partial}{\partial t}(t^n) \cdot e^{-r^2/4t} + t^n \cdot \frac{\partial}{\partial t}\left(e^{-r^2/4t}\right)$$

$$= n t^{n-1} e^{-r^2/4t} + t^n \cdot e^{-r^2/4t} \cdot \frac{\partial}{\partial t}\left(\frac{-r^2}{4t}\right)$$

$$= n t^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t} \left(\frac{-r^2}{4}\right)\left(\frac{-1}{t^2}\right)$$

$$\text{LHS} = \frac{\partial\theta}{\partial t} = e^{-r^2/4t}\left[n t^{n-1} + \frac{r^2}{4} t^{n-2}\right]$$

Now $\theta = t^n e^{-r^2/4t}$

differentiating wrt r

$$\frac{\partial\theta}{\partial r} = t^n \cdot e^{-r^2/4t} \cdot \frac{\partial}{\partial r}\left(\frac{-r^2}{4t}\right) = t^n e^{-r^2/4t}\left(\frac{-2r}{4t}\right)$$

$$= -\frac{1}{2} t^{n-1} \cdot r \, e^{-r^2/4t}$$

$$r^2 \frac{\partial\theta}{\partial r} = -\frac{1}{2} t^{n-1} r^3 e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial\theta}{\partial r}\right) = -\frac{1}{2} t^{n-1}\left\{\frac{\partial}{\partial r}(r^3) \cdot e^{-r^2/4t} + r^3 \frac{\partial}{\partial r}\left(e^{-r^2/4t}\right)\right\}$$

$$= -\frac{1}{2}t^{n-1}\left\{3r^2 e^{-r^2/4t} + r^3 e^{-r^2/4t}\cdot\frac{\partial}{\partial r}\left(\frac{-r^2}{4t}\right)\right\}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{1}{2}t^{n-1}\left\{3r^2 e^{-r^2/4t} - \frac{r^4}{2t}e^{-r^2/4t}\right\}$$

$$\text{RHS} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{1}{2}t^{n-1}\left\{3e^{-r^2/4t} - \frac{r^2}{2t}e^{-r^2/4t}\right\}$$

$$\text{RHS.} = e^{-r^2/4t}\left\{-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right\}$$

Comparing LHS & RHS.

$$e^{-r^2/4t}\left\{nt^{n-1} + \frac{r^2}{4}t^{n-2}\right\} = e^{-r^2/4t}\left\{-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right\}$$

By comparing, we get $n = -\frac{3}{2}$.

5. If $z = x^y + y^x$, verify that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$.

$\frac{d}{dx}(a^x) = a^x \log a$

$\frac{d}{dx}(y^n) = ny^{n-1}$

Soln :- $z = x^y + y^x$

Differentiating wrt y first

$$\frac{\partial z}{\partial y} = x^y \log x + x y^{x-1}$$

differentiating $\frac{\partial z}{\partial y}$ wrt x

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(x^y)\log x + x^y\frac{\partial}{\partial x}(\log x) + \frac{\partial}{\partial x}(x)y^{x-1} + x\frac{\partial}{\partial x}(y^{x-1})$$

$$= yx^{y-1}\log x + x^y\cdot\frac{1}{x} + y^{x-1} + xy^{x-1}\log y$$

$$\text{LHS} = yx^{y-1}\log x + x^{y-1} + y^{x-1} + xy^{x-1}\log y$$

LHS $= yx \log x + x + y + \ldots$

To calculate RHS,

H.W.

$$\frac{\partial z}{\partial x} =$$

$$\frac{\partial^2 z}{\partial y \partial x} =$$

1/10/2022 01:15 PM

6. If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ *and* $a^2 + b^2 + c^2 = 1$, Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Soln:- $u = 3(ax+by+cz)^2 - (x^2+y^2+z^2)$

differentiating wrt x twice partially

$$\frac{\partial u}{\partial x} = 6(ax+by+cz)\cdot(a) - 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 6a(a) - 2 = 6a^2 - 2 \quad \text{—— (1)}$$

differentiating wrt y twice

$$\frac{\partial u}{\partial y} = 6(ax+by+cz)\cdot(b) - 2y$$

$$\frac{\partial^2 u}{\partial y^2} = 6(b)\cdot(b) - 2 = 6b^2 - 2 \quad \text{—— (2)}$$

differentiating u wrt z twice

$$\frac{\partial u}{\partial z} = 6(ax+by+cz)\cdot(c) - 2z$$

$$\frac{\partial^2 u}{\partial z^2} = 6c(c) - 2 = 6c^2 - 2 \quad \text{—— (3)}$$

(1) + (2) + (3) =>

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6(a^2+b^2+c^2) - 6$$

Now $a^2+b^2+c^2 = 1$

$$\frac{\partial^2 u}{\ldots} + \frac{\partial^2 u}{\ldots} + \frac{\partial^2 u}{\ldots} = 6(1) - 6 = 0$$ Hence proved.

Now

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6(1) - 6 = 0 \qquad \text{Hence proved.}$$

7. If $u = f(r), r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$.

Soln :- $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$

differentiating u wrt x

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r} = \frac{f'(r) \cdot x}{r}$$

$r^2 = x^2 + y^2 + z^2$

diff wrt x

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{2x}{2r}$$

$$\Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

Similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

differentiating wrt x again

$$\frac{\partial^2 u}{\partial x^2} = \frac{r \cdot \frac{\partial}{\partial x}\left(f'(r) \cdot x\right) - f'(r) \cdot x \frac{\partial}{\partial x}(r)}{r^2}$$

$$= \frac{r \cdot \left[f'(r) \cdot \frac{\partial x}{\partial x} + \frac{\partial}{\partial x}\left[f'(r)\right] \cdot x \right] - f'(r) \cdot x \left(\frac{x}{r}\right)}{r^2}$$

$$= \frac{r\left[f'(r)(1) + f''(r) \cdot \frac{\partial r}{\partial x} \cdot x \right] - f'(r) \cdot \frac{x^2}{r}}{r^2}$$

$$= \frac{r\left[f'(r) + f''(r) \cdot \frac{x^2}{r} \right] - f'(r) \cdot \frac{x^2}{r}}{r^2}$$

$$= \frac{r f'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r)}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(r^2 - x^2) f'(r) + x^2 r f''(r)}{\ } \quad \text{— (1)}$$

r^3

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{(r^2 - y^2) f'(r) + y^2 r f''(r)}{r^3}$ —— 2

$\frac{\partial^2 u}{\partial z^2} = \frac{(r^2 - z^2) f'(r) + z^2 r f''(r)}{r^3}$ —— 3

1 + 2 + 3

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{[3r^2 - (x^2+y^2+z^2)] f'(r) + r f''(r)(x^2+y^2+z^2)}{r^3}$$

$$= \frac{(3r^2 - r^2) f'(r) + r f''(r) \cdot r^2}{r^3}$$

$$= \frac{2r^2}{r^3} f'(r) + \frac{r^3}{r^3} f''(r)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{r} f'(r) + f''(r)$$

8. If $u = e^{x^2+y^2+z^2}$, prove that $\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = 8\, xyzu$.

Soln :-

$$u = e^{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial z} = e^{x^2+y^2+z^2} \cdot \frac{\partial}{\partial z}(x^2+y^2+z^2) = 2z\, e^{x^2+y^2+z^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial y \partial z} = 2z \cdot \left[e^{x^2+y^2+z^2} \cdot \frac{\partial}{\partial y}(x^2+y^2+z^2)\right]$$

$$= 2z \left[e^{x^2+y^2+z^2} \cdot 2y\right]$$

$x^2 u$ $\quad$ $x^2+y^2+z^2$

$$\frac{\partial^2 u}{\partial y \partial z} = 4yz\, e^{x^2+y^2+z^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial^3 u}{\partial x \partial y \partial z} = 4yz\left[e^{x^2+y^2+z^2} \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)\right]$$

$$= 4yz\left[e^{x^2+y^2+z^2} \cdot 2x\right]$$

$$= 8xyz\, e^{x^2+y^2+z^2}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = 8xyzu$$

9. If $z = u(x,y)\, e^{ax+by}$ where $u(x,y)$ is such that $\frac{\partial^2 u}{\partial x \partial y} = 0$, find the constants a, b such that $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$.

Soln:- $z = u(x,y)\, e^{ax+by}$

differentiating wrt x partially

$$\frac{\partial z}{\partial x} = u(x,y)\frac{\partial}{\partial x}\left(e^{ax+by}\right) + \frac{\partial}{\partial x}\left(u(x,y)\right) e^{ax+by}$$

$$= u(x,y) \cdot e^{ax+by} \cdot (a) + \frac{\partial u}{\partial x} e^{ax+by}$$

$$\frac{\partial z}{\partial x} = e^{ax+by}\left(au + \frac{\partial u}{\partial x}\right) \quad \text{——} \quad (1)$$

differentiating z wrt y

$$z = u(x,y)\, e^{ax+by}$$

$$\frac{\partial z}{\partial y} = u(x,y)\frac{\partial}{\partial y}\left(e^{ax+by}\right) + \frac{\partial}{\partial y}\left(u(x,y)\right) \cdot e^{ax+by}$$

$$= u(x,y) \cdot e^{ax+by} \cdot (b) + \frac{\partial u}{\partial y} e^{ax+by}$$

$$\frac{\partial z}{\partial y} = e^{ax+by}\left[bu + \frac{\partial u}{\partial y}\right] \quad \text{---} \; (2)$$

differentiating $\frac{\partial z}{\partial y}$ partially wrt x

$$\frac{\partial^2 z}{\partial x \partial y} = e^{ax+by} \cdot \frac{\partial}{\partial x}\left[bu + \frac{\partial u}{\partial y}\right] + \left[bu + \frac{\partial u}{\partial y}\right] \cdot \frac{\partial}{\partial x}\left(e^{ax+by}\right)$$

$$= e^{ax+by}\left[b \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y}\right] + \left[bu + \frac{\partial u}{\partial y}\right] \cdot e^{ax+by} \cdot (a)$$

$$= e^{ax+by}\left[abu + a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y}\right]$$

but $\frac{\partial^2 u}{\partial x \partial y} = 0$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = e^{ax+by}\left[abu + a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x}\right] \quad \text{---} \; (3)$$

Now we are given that

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$$

using (1), (2), (3)

$$e^{ax+by}\left[abu + a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x}\right] - e^{ax+by}\left[au + \frac{\partial u}{\partial x}\right]$$
$$- e^{ax+by}\left[bu + \frac{\partial u}{\partial y}\right] + e^{ax+by} \cdot u = 0$$

$$e^{ax+by}\left[\underline{abu} + a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x} - \underline{au} - \frac{\partial u}{\partial x} - \underline{bu} - \frac{\partial u}{\partial y} + \underline{u}\right] = 0$$

$$e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + (a-1)(b-1)u\right] = 0$$

$$e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y}+(b-1)\frac{\partial u}{\partial x}+(a-1)(b-1)u\right]=0$$

$$e^{ax+by}\neq 0,\quad \frac{\partial u}{\partial x}\neq 0,\quad \frac{\partial u}{\partial y}\neq 0,\quad u=0$$

$$\Rightarrow a-1=0,\quad b-1=0 \Rightarrow a=1,\ b=1.$$

10. If $a^2x^2+b^2y^2=c^2z^2$, evaluate $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2}+\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$

Soln:- $c^2z^2=a^2x^2+b^2y^2 \Rightarrow z^2=\frac{a^2}{c^2}x^2+\frac{b^2}{c^2}y^2$

differentiating wrt x

$$2z\cdot\frac{\partial z}{\partial x}=\frac{a^2}{c^2}(2x)+0$$

$$\frac{\partial z}{\partial x}=\frac{a^2}{c^2}\cdot\frac{x}{z}$$

Differentiating $\frac{\partial z}{\partial x}$ wrt x partially

$$\frac{\partial^2 z}{\partial x^2}=\frac{a^2}{c^2}\ \frac{z\cdot\frac{\partial}{\partial x}(x)-x\cdot\frac{\partial}{\partial x}(z)}{z^2}$$

$$=\frac{a^2}{c^2}\cdot\frac{z\cdot(1)-x\cdot\frac{\partial z}{\partial x}}{z^2}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{a^2}{c^2}\left[\frac{z-x\cdot\frac{a^2}{c^2}\left(\frac{x}{z}\right)}{z^2}\right]$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{a^2}{c^2}\left[\frac{c^2z^2-a^2x^2}{c^2z^3}\right]$$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^4}\left[\frac{c^2z^2 - a^2x^2}{z^3}\right] \quad \text{—— } 1$$

Similarly,

$$\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^4}\left[\frac{c^2z^2 - b^2y^2}{z^3}\right] \quad \text{—— } 2$$

1 + 2

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^4}\left[\frac{2c^2z^2 - (a^2x^2 + b^2y^2)}{z^3}\right]$$

$$= \frac{1}{c^4}\left[\frac{2c^2z^2 - c^2z^2}{z^3}\right]$$

$$= \frac{1}{c^4}\left[\frac{c^2z^2}{z^3}\right]$$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2 z}$$