Minimal Polynomial

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$f(x) = x^{10} - 9x^7 + 6x^3 - x^2 + 1$

$f(A) = A^{10} - 9A^7 + 6A^3 - A^2 + I = 0$

MINIMAL POLYNOMIAL AND MINIMAL EQUATION OF A MATRIX

Let $f(x)$ be a polynomial in x and A be a square matrix of order n.

If $f(A) = 0$ then we say that $f(x)$ **annihilates the matrix** A.

We know that by Caley- Hamilton theorem every matrix satisfies its characteristic equation.

Hence, the characteristic polynomial of the matrix A annihilates A.

$\lambda^3 - 3\lambda^2 + 5\lambda - 6 = 0$

$A^3 - 3A^2 + 5A - 6I = 0$

MONIC POLYNOMIAL:

A polynomial in x, in which the coefficient of the highest power of x is unity is called a **monic polynomial**.

Thus, $x^3 + 2x^2 + 3x - 7$ is a monic polynomial while $2x^3 - 3x^2 + 4x - 9$ is not a **monic polynomial**.

MINIMAL POLYNOMIAL OF A MATRIX:

The monic polynomial of lowest degree that annihilates a matrix A is called **minimal polynomial** of A.

Further, if $f(x)$ is the minimal polynomial of A then the equation $f(x) = 0$ is called the **minimal equation** of the matrix A.

$A \to 3\times3$

deg - 3 $\to$ deg 2 — deg 1

λ^3 - - - - - - characteristic poly

If a matrix is of order n then its characteristic polynomial is of degree n.

We know that the characteristic polynomial of A annihilates A. Hence, the degree of minimal polynomial of A cannot be greater than n.

Note: **(i)** Minimal polynomial of a matrix is unique

(ii) Minimal polynomial of a matrix is a divisor of every polynomial that annihilates this matrix

(iii) Minimal polynomial of a matrix is a divisor of the characteristic polynomial of that matrix

all the roots of minimal poly = some roots of ch. poly.

(iv) Null matrix is only matrix whose minimal polynomial is x

$f(x) = x$, $f(0) = 0$ ✓

(v) Unit matrix is the only matrix whose minimal polynomial is $(x - 1)$

Identity

$f(x) = x - I$, $f(I) = I - I = 0$. ✓

DEROGATORY AND NON – DEROGATORY MATRICES:

An n – rowed square matrix is said to be **derogatory or Non – derogatory** according as the degree of its

Identity $f(I) = I$

DEROGATORY AND NON – DEROGATORY MATRICES:

An n – rowed square matrix is said to be **derogatory or Non – derogatory** according as the degree of its minimal equation is less than or equal to n.

A

$\downarrow$

Minimal poly.

deg = n (ch. poly = min poly) $\rightarrow$ Non derogatory

deg $< n$ $\downarrow$ Derogatory

(1) If all the eigen values are distinct then characteristic polynomial is equal to the minimal polynomial and the matrix is Non-derogatory

(2) If there are repeated eigen values, then we need to check whether we can find a lower deg. polynomial which annihilates the matrix and then decide about the minimal poly.

SOME SOLVED EXAMPLES:

1. Show that the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is derogatory

Sol^n:- Ch. eq^n of A is $\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$\therefore$ Eigen values of A are $\lambda = 1, 2, 2$

let us now find the minimal polynomial of A.

we know that each eigen value of A is a root of minimum polynomial of A.

So if $f(x)$ is the minimal polynomial of A then $(x-1)$ and $(x-2)$ are the facters of $f(x)$

let us see whether the polynomial

$(x-1)(x-2) = x^2 - 3x + 2$ annhilates A.

ie to check $A^2 - 3A + 2I = 0$

$$A^2 = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}^2 = \begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix}$$

$$\therefore A^2 - 3A + 2I = \begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix} - 3\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 - 3A + 2I = O.$$

$\therefore f(x) = x^2 - 3x + 2$ annhilates A.

Thus $f(x)$ is the monic polynomial of lowest degree that annhilates A. Hence $f(x)$ is the minimal polynomial of A.

Since its degree is less than the order of A, A is derogatory.

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2. Show that the matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ is non- derogatory

Sol^n :- ch. eq^n of A is $\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$\therefore$ Eigen values of A are $\lambda = -2, 1, 3$

All the eigen values are distinct.

$\therefore$ $f(x) = (x+2)(x-1)(x-3)$ is the minimal polynomial

$\therefore$ deg of minimal polynomial is equal to the order of the matrix. Hence the matrix is non- derogatory.

Q.3 :- $A = \begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$$

$$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$$

Eigen values $\lambda = 2, \underline{2}, \underline{4}$

$$f(x) = (x-2)(x-4) = x^2 - 6x + 8$$

check whether $f(x)$ annhilates A.

check $A^2 - 6A + 8I = 0$

$$A^2 - 6A + 8I = \begin{bmatrix} 4 & -18 & 18 \\ 0 & 10 & -6 \\ 0 & -6 & 10 \end{bmatrix} - 6\begin{bmatrix} 2 & -3 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore f(x) = x^2 - 6x + 8$ is the minimal polynomial

deg $f(x) <$ order of A

$\therefore$ A is derogatory.

Ex:- Find the Symmetric matrix $A_{3\times3}$ having the eigen values $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = 15$ with the corresponding eigen vectors $X_1 = [1, 2, 2]'$, $X_2 = [-2, -1, 2]'$ and X_3.

Soln:- let $X_3 = [x_1, x_2, x_3]'$ be the third eigen vector corresponding to eigen value $\lambda = 15$.

Since the required matrix A is symmetric and all eigenvalues are distinct, the three eigenvectors corresponding to 3 eigenvalues are orthogonal.

X_3 is orthogonal to X_1 & X_2

$$X_1 \cdot X_3 = 0 \Rightarrow [1 \ 2 \ 2] \cdot [x_1 \ x_2 \ x_3] = 0$$

$$\Rightarrow x_1 + 2x_2 + 2x_3 = 0$$

$$X_2 \cdot X_3 = 0 \Rightarrow [-2, -1, 2] \cdot [x_1, x_2, x_3] = 0$$

$$\Rightarrow -2x_1 - x_2 + 2x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{-x_2}{6} = \frac{x_3}{3}$$

$\therefore \ X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvecter corresponding to $\lambda = 15$

Since A is symmetric, it is orthogonally similar to a diagonal matrix D.

There exists an orthogonal matrix P such that

$$P^{-1}AP = D \quad \text{ie} \quad A = PDP^{-1} = PDP^t$$

$$(P \text{ is orthogonal} \Rightarrow P^{-1} = P^t)$$

Since P is an orthogonal matrix, we divide each vector by its norm $\quad X_1 = [1 \ 2 \ 2]$

$$\|X_1\| = \sqrt{1+4+4} = 3$$

$$\|X_2\| = \sqrt{4+1+4} = 3$$

$$\|x_3\| = \sqrt{4+4+1} = 3$$

$$\therefore P = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$A = PDP^t = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Note : $A = MDM^{-1}$ $\quad M = [x_1 \ \ x_2 \ \ x_3]$

M^{-1} ?