Function of Square matrices

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CALCULATION OF POWERS OF MATRIX (FUNCTIONS OF SQUARE MATRIX):

If A is a non – singular square matrix with distinct Eigen values then we can find any power of A. i.e A^k(k is a positive integer) by the process explained below.

we have $M^{-1}AM = D$

Operating by M on the left and by M^{-1} on the right

$MM^{-1}AMM^{-1} = MDM^{-1}$

$\therefore (MM^{-1})A(MM^{-1}) = MDM^{-1}$

$\therefore A = MDM^{-1}$

$\therefore A^n = (MDM^{-1})(MDM^{-1}) \dots\dots\dots (MDM^{-1})\ (n\ times)$

$\therefore A^n$

$= MD(M^{-1}M)D(M^{-1}M) \dots\dots\dots (M^{-1}M)DM^{-1}$

$= MD \dots\dots\dots DM^{-1}$

$$= MD^nM^{-1} = M\begin{bmatrix} \lambda_1^n & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_n^n \end{bmatrix}M^{-1}$$

Note: Above method can be applied for any function of A i.e. $\boxed{f(A) = M\,f(D)\,M^{-1}}$

$M^{-1}AM = D$ (Diagonalisation)

$M = [X1\ \ X2\ \ X3]$

$$D = \begin{bmatrix} \lambda 1 & 0 & 0 \\ 0 & \lambda 2 & 0 \\ 0 & 0 & \lambda 3 \end{bmatrix}$$

$A = MDM^{-1}$

$A^n = MD^nM^{-1}$

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$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad D^{10} = \begin{bmatrix} 1^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 3^{10} \end{bmatrix}$$

$f(A) = M\,f(D)\,M^{-1}$

$\cos A = M \cos D\, M^{-1}$

$$\cos D = \begin{bmatrix} \cos 1 & 0 & 0 \\ 0 & \cos 2 & 0 \\ 0 & 0 & \cos 3 \end{bmatrix}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$\cos D = I - \frac{1}{2!}D^2 + \frac{1}{4!}D^4 - \frac{1}{6!}D^6 + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2!}\begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix} + \frac{1}{4!}\begin{bmatrix} \alpha^4 & 0 \\ 0 & \beta^4 \end{bmatrix} - \frac{1}{6!}\begin{bmatrix} \alpha^6 & 0 \\ 0 & \beta^6 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \dots & 0 \\ 0 & 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} + \dots \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 \\ 0 & \cos\beta \end{bmatrix}$$

ANOTHER METHOD: (General method) can be applied to any square matrix.

① If A is 2×2 matrix, we write

$f(A) = \alpha_1 A + \alpha_0 I$ and find α_1 & α_0 using the eigen values. of A

② If A is 3×3 matrix, we write

$f(A) = \underline{\alpha_2} A^2 + \underline{\alpha_1} A + \underline{\alpha_0} I$ and find α_2, α_1 & α_0 using the eigen values of A.

$A^{50} \longrightarrow$ divide by the ch. poly

$A^{50} = (\underbrace{\text{divisor}}_{=0} \times \text{quotient}) + \underline{\text{Remainder}}$

$A^{50} = \text{Remainder} = \begin{cases} \alpha_1 A + \alpha_0 I & (A \text{ is } 2\times 2) \\ \alpha_2 A^2 + \alpha_1 A + \alpha_0 I & (A \text{ is } 3\times 3) \end{cases}$

SOME SOLVED EXAMPLES:

1. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find A^{50}

Soln:- Ch. eqn of A is $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

(distinct)

(distinct)

$\therefore$ eigen values of A are $\lambda = 1, 3$.

Find eigen vectors now.

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$$

Let $x_1 = t \Rightarrow x_2 = -t$

$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector for $\lambda = 1$

For $\lambda = 3$, $[A - \lambda I]X = 0$ $[A - 3I]X = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector for $\lambda = 3$.

$\therefore$ Modal matrix $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ & $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Now $f(A) = M f(D) M^{-1}$

$$A^{50} = M D^{50} M^{-1}$$

$$\therefore A^{50} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3^{50} \\ -1 & 3^{50} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$M^{-1} = \frac{\operatorname{adj} M}{|M|}$

$|M| = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$

$\operatorname{adj} M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$M^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} \cdots & \cdots \\ -1 & 3^{50} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} \cdots & \cdots \\ 1 & 1 \end{bmatrix}$$

$$A^{50} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$$

Now solving by method - II

let $A^{50} = \alpha_1 A + \alpha_0 I$ — (1) (A is 2×2 matrix)

we assume that this relation is true for λ

$\lambda^{50} = \alpha_1 \lambda + \alpha_0$ — (2)

for $\lambda = 1$, $(1)^{50} = \alpha_1(1) + \alpha_0 \Rightarrow \alpha_1 + \alpha_0 = 1$ — (3)

for $\lambda = 3$, $(3)^{50} = \alpha_1(3) + \alpha_0 \Rightarrow 3\alpha_1 + \alpha_0 = 3^{50}$ — (4)

(4) - (3) $\Rightarrow 2\alpha_1 = 3^{50} - 1 \Rightarrow \alpha_1 = \frac{3^{50}-1}{2}$

Sub in (3)

$\alpha_1 + \alpha_0 = 1 \Rightarrow \alpha_0 = 1 - \alpha_1 = 1 - \frac{3^{50}-1}{2}$

$= \frac{3-3^{50}}{2}$

Sub. α_0 & α_1 in (1)

$$A^{50} = \alpha_1 A + \alpha_0 I = \frac{3^{50}-1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{3-3^{50}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{50}-1+\frac{3-3^{50}}{2} & \frac{3^{50}-1}{2} \\ \frac{3^{50}-1}{2} & 3^{50}-1+\frac{3-3^{50}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3^{50}+1 & \frac{3^{50}-1}{2} \\ \cdots & \cdots \end{bmatrix} = \frac{1}{\cdots}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ \cdots & \cdots \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} \frac{3^{50}+1}{2} & \frac{3^{50}-1}{2} \\ \frac{3^{50}-1}{2} & \frac{3^{50}+1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$$

2. If $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$, prove that $A^{50} = \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$

Soln:- Ch. eqn of A is $\begin{vmatrix} 2-\lambda & 3 \\ -3 & -4-\lambda \end{vmatrix} = 0$

$(2-\lambda)(-4-\lambda) + 9 = 0$

$-8 - 2\lambda + 4\lambda + \lambda^2 + 9 = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda = -1, -1$ (repeated).

we use method - 2 here.

let $f(A) = A^{50} = \alpha_1 A + \alpha_0 I$ —— (1)

writing in terms of λ

$f(\lambda) = \lambda^{50} = \alpha_1 \lambda + \alpha_0$ —— (2)

put $\lambda = -1$, $(-1)^{50} = \alpha_1(-1) + \alpha_0$

$-\alpha_1 + \alpha_0 = 1$ —— (3)

differentiate eqn (2) wrt λ.

$50\lambda^{49} = \alpha_1$

put $\lambda = -1$, $50(-1)^{49} = \alpha_1$ $\Rightarrow$ $\boxed{\alpha_1 = -50}$

Sub in (3), $-\alpha_1 + \alpha_0 = 1$ $\Rightarrow \alpha_0 = 1 + \alpha_1 = -49$

$\boxed{\alpha_0 = -49}$

Sub α_0, α_1 in (1),

$$A^{50} = \alpha_1 A + \alpha_0 I = -50A - 49I$$

$$= -50\begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$$

3. Find e^A and 4^A if $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

Soln :- Ch. eqn of A is $\begin{vmatrix} \frac{3}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2}-\lambda \end{vmatrix} = 0$

$$\left(\frac{3}{2}-\lambda\right)^2 - \frac{1}{4} = 0$$

$$\frac{9}{4} - 3\lambda + \lambda^2 - \frac{1}{4} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = 1, 2$$

we will use method-2.

let $f(A) = e^A = \alpha_1 A + \alpha_0 I$ —— (1)

writting in terms of λ

$$e^\lambda = \alpha_1 \lambda + \alpha_0$$

put $\lambda = 1$, $e = \alpha_1 + \alpha_0$ —— (2)

put $\lambda = 2$, $e^2 = 2\alpha_1 + \alpha_0$ —— (3)

(3) − (2) $\Rightarrow$ $\boxed{\alpha_1 = e^2 - e}$

$e^2 - e + \alpha_0$ ——

$③ - ②$

Sub in ② $\Rightarrow$ $e = e^2 - e + \alpha_0$

$$\Rightarrow \boxed{\alpha_0 = 2e - e^2}$$

Sub in ① $e^A = \alpha_1 A + \alpha_0 I$

$$= (e^2 - e)\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} + (2e - e^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3(e^2-e)}{2} + 2e - e^2 & \frac{e^2-e}{2} \\ \frac{e^2-e}{2} & \frac{3}{2}(e^2-e) + 2e - e^2 \end{bmatrix}$$

$$e^A = \begin{bmatrix} \frac{e^2+e}{2} & \frac{e^2-e}{2} \\ \frac{e^2-e}{2} & \frac{e^2+e}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^2+e & e^2-e \\ e^2-e & e^2+e \end{bmatrix}$$

Replacing e by 4

$$4^A = \frac{1}{2}\begin{bmatrix} 4^2+4 & 4^2-4 \\ 4^2-4 & 4^2+4 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

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4. If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ then prove that $3 \tan A = A \tan 3$

Soln :- Ch. eqn of A is $\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 9 = 0$$

Eigen values $\rightarrow$ $\lambda = \pm 3$

Let $f(A) = \tan A = \alpha_1 A + \alpha_0 I$ —— (1)

writing in terms of λ

$\tan\lambda = \alpha_1 \lambda + \alpha_0$ —— (2)

put $\lambda = 3$, $\tan 3 = 3\alpha_1 + \alpha_0$ —— (3)

put $\lambda = -3$, $\tan(-3) = -3\alpha_1 + \alpha_0$

$-\tan 3 = -3\alpha_1 + \alpha_0$ —— (4)

Adding (3) & (4) $\Rightarrow$ $2\alpha_0 = 0$ $\Rightarrow$ $\boxed{\alpha_0 = 0}$

Sub $\alpha_0 = 0$ in (3) $\Rightarrow$ $\tan 3 = 3\alpha_1$

$\Rightarrow \boxed{\alpha_1 = \frac{1}{3}\tan 3}$

Sub α_1 & α_0 in (1)

$\tan A = \alpha_1 A + \alpha_0 I = \left(\frac{1}{3}\tan 3\right) A + 0 I$

$\Rightarrow$ $3\tan A = (\tan 3) A$

$\Rightarrow$ $3\tan A = A \tan 3$ Hence proved.

5. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{50}

5. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find A^{50}

Sol^n :- Ch. eq^n of A is $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\left[\lambda^2-1\right] = 0$$

$$\lambda^2 - 1 - \lambda^3 + \lambda = 0$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Eigen values are $\lambda = -1, 1, 1$

Let $f(A) = A^{50} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$ —— (1)

writing in terms of λ

$$\lambda^{50} = \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 \quad \text{—— (2)}$$

put $\lambda = -1$, $(-1)^{50} = \alpha_2(-1)^2 + \alpha_1(-1) + \alpha_0$

$$1 = \alpha_2 - \alpha_1 + \alpha_0 \quad \text{—— (3)}$$

put $\lambda = 1$, $(1)^{50} = \alpha_2(1)^2 + \alpha_1(1) + \alpha_0$

$$1 = \alpha_2 + \alpha_1 + \alpha_0 \quad \text{—— (4)}$$

diff. (2) wrt λ

$$50\lambda^{49} = 2\alpha_2\lambda + \alpha_1$$

put $\lambda = 1$, $50(1)^{49} = 2\alpha_2(1) + \alpha_1$

$$50 = 2\alpha_2 + \alpha_1 \quad \text{--- (5)}$$

Solving (3), (4) & (5) we get.

$$\alpha_2 = 25, \quad \alpha_1 = 0, \quad \alpha_0 = -24$$

Sub. $\alpha_0, \alpha_1, \alpha_2$ in eq^n (1)

$$A^{50} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I = 25A^2 - 24I$$

$$A^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

6. Show that $\cos 0_{3\times3} = I_{3\times3}$

Sol^n :- $O_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{vmatrix} 0-\lambda & 0 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0$

$$\lambda^3 = 0.$$

Eigen values of $O_{3\times3}$ are $\lambda = 0, 0, 0.$

Let $\cos O_{3\times3} = \alpha_2 O_{3\times3}^2 + \alpha_1 O_{3\times3} + \alpha_0 O_{3\times3}$ --- (1)

writing in terms of λ

$$\cos\lambda = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 \quad \text{--- (2)}$$

put $\lambda = 0$, $\cos 0 = \alpha_2(0)^2 + \alpha_1(0) + \alpha_0$

$$\boxed{1 = \alpha_0}$$

differentiating (2) wrt λ

differentiating (2) wrt λ

$$-\sin\lambda = 2\alpha_2\lambda + \alpha_1 \quad \text{---} \quad (3)$$

put $\lambda = 0$, $\quad -\sin 0 = 2\alpha_2(0) + \alpha_1$

$$\Rightarrow \boxed{\alpha_1 = 0}$$

differentiating (3) wrt λ

$$-\cos\lambda = 2\alpha_2$$

put $\lambda = 0$, $\quad -\cos 0 = 2\alpha_2$

$$\Rightarrow \boxed{\alpha_2 = -\frac{1}{2}}$$

sub $\alpha_0, \alpha_1, \alpha_2$ in eqn (1)

$$\cos O_{3\times3} = \alpha_2 O_{3\times3}^2 + \alpha_1 O_{3\times3} + \alpha_0 I$$

$$\cos O_{3\times3} = \alpha_0 I$$

$$\therefore \boxed{\cos O_{3\times3} = I}$$

<u>H.W</u> prove that $\sin O_{3\times3} = O_{3\times3}$.