Properties of Eigen Values and Eigen Vectors

Monday, December 27, 2021 1:33 PM

Theorem 1: Prove that zero is an eigenvalues of a matrix A if and only if A is singular.

Theorem 2: Prove that the matrix A and its transpose A^T has same characteristic roots.

Theorem 3: Prove that the eigenvalues of a diagonal matrix are precisely the diagonal elements.

Theorem 4: Prove that the eigenvalues of a triangular matrix are precisely the diagonal elements.

Theorem 5: λ is an Eigen value of the matrix A if and only if there exists a non – zero vector X such that ~~$AX = \lambda X$~~. $AX = \lambda X$

Theorem 6: If X is an Eigen vector of a matrix A corresponding to an Eigen value λ then kX (k is a non – zero scalar) is also an Eigen vector of A corresponding to the same Eigen value λ.

Theorem 7: (Uniqueness of Eigen Value):
If X is an Eigen vector of a matrix A then X cannot correspond to more than one Eigen values of A.

Theorem 8: (Linear independence of Eigen Vectors):
Eigen vectors corresponding to distinct Eigen values of a matrix are linearly independent.

Theorem 9: Eigen values of a Hermitian matrix are real. $|A|$ = product of eigenvalues

Corollary 1: The determinant of a Hermitian matrix is real.

Corollary 2: Eigen values of a real symmetric matrix are all real. A^t A^θ $\bar{A}$

Corollary 3: Eigen values of a Skew – Hermitian matrix are either purely imaginary or zero.

Corollary 4: The Eigen values of a real skew – symmetric matrix are purely imaginary or zero.

Theorem 10: The Eigen values of unitary matrix are of unit modulus. (have absolute value one).

Corollary: Eigen values of an orthogonal matrix are of unit modulus.

* **Theorem 11:** The Eigen vectors corresponding to distinct Eigen values of a real symmetric matrix are orthogonal. $x_1 \cdot x_2 = 0.$

Theorem 12: Any two Eigen vectors corresponding to two distinct Eigen values of a unitary matrix are orthogonal.

$x_1 \quad x_2 \quad x_3$

$A \to 1, 2, 3$

$3A \to 3, 6, 9$

$-5A \to -5, -10, -15$

$A^{-1} \to \frac{1}{1}, \frac{1}{2}, \frac{1}{3}$

$A^2 \to 1^2, 2^2, 3^2$

$1, 4, 9$

$A^3 \to 1, 8, 27$

Note:

1. If one Eigen value of a matrix A is $a + ib$ then another eigen value must be $a - ib$.
2. If λ is an Eigen value of A then $\bar{\lambda}$ is an eigen value of A^θ
3. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A then show that $k\lambda_1, k\lambda_2, \ldots\ldots k\lambda_n$ are the Eigen value kA.
4. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A then show that $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots\ldots \frac{1}{\lambda_n}$ are the Eigen values of A^{-1}
5. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A then show that $\lambda_1^2, \lambda_2^2, \ldots\ldots, \lambda_n^2$ are the eigen values of A^2.
6. If λ is an Eigen value of a non – singular matrix A, prove that $\frac{|A|}{\lambda}$ is an eigen value of adj A.
7. If λ is an Eigen value of the matrix A then $\lambda \pm k$ is an eigen value of $A \pm kI$
8. If $f(x)$ is an algebraic polynomial in x and λ is an Eigen value and X is the corresponding Eigen vector of a square matrix A then $f()$ is an eigen value and X is the corresponding eigen vector of $f(A)$.

Solved Examples

1. If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ Where a, b, c are positive integers, then prove that

(i) a + b+ c is an Eigen value of A and **(ii)** ~~if A is non – singular,~~ one of the Eigen values is negative.

The characteristic eqn of A is

$|a-\lambda \quad b \quad c \;|$

The characteristic eqⁿ of A is

$$\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$$

By $C_1 + C_2 + C_3$

$$\begin{vmatrix} a+b+c-\lambda & b & c \\ a+b+c-\lambda & c-\lambda & a \\ a+b+c-\lambda & a & b-\lambda \end{vmatrix} = 0$$

$$\underline{(a+b+c-\lambda)} \begin{vmatrix} 1 & b & c \\ 1 & c-\lambda & a \\ 1 & a & b-\lambda \end{vmatrix} = 0$$

$\therefore$ $a+b+c-\lambda = 0 \Rightarrow \lambda = a+b+c$.

$\therefore$ one of the eigen values is $a+b+c$

If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A then

$\underline{\lambda_1 + \lambda_2 + \lambda_3 = \text{trace of } A} = a+b+c$.

but one of the eigen values $= a+b+c$

say $\lambda_1 = a+b+c$

$\Rightarrow \lambda_2 + \lambda_3 = 0$

As A is non singular, λ_2, λ_3 are not zero.

$\Rightarrow$ either of λ_2, or λ_3 must be negative.

2. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of a 3×3 matrix then prove that the Eigen values of $adj\ A$ are $\lambda_1\lambda_2, \lambda_2\lambda_3$ and $\lambda_3\lambda_1$

Soln :- $\lambda_1, \lambda_2, \lambda_3$ are eigen values of A

$\Rightarrow |A| = \lambda_1 \lambda_2 \lambda_3$

$\Rightarrow$ eigen values of adj A $\rightarrow \frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$

$\rightarrow \frac{\lambda_1\lambda_2\lambda_3}{\lambda_1}, \frac{\lambda_1\lambda_2\lambda_3}{\lambda_2}, \frac{\lambda_1\lambda_2\lambda_3}{\lambda_3}$

$\rightarrow \lambda_2\lambda_3, \lambda_1\lambda_3, \lambda_1\lambda_2$

$2, 3, 4 \rightarrow A \qquad \rightarrow |A| = 2 \times 3 \times 4 = 24$

$adj\ A \rightarrow \frac{|A|}{2}, \frac{|A|}{3}, \frac{|A|}{4} = 12, 8, 6$

12/29/2021 2:23 PM

3. If A is a real square matrix of order n where n is an **odd** positive integer, then show that A has at least one real Eigen value.

Soln :- Since A is a matrix of odd order, it has characteristic eqn of degree n which is odd.

Case-I : The char. eqn has all real roots

Then we get the req. answer.

Case-II : The char. eqn has complex roots

but the complex roots always occur in pairs

$\therefore$ no. of complex eigen values will always be even.

$\therefore$ atleast one eigen values would be new.

4. The sum of the Eigen values of a 3×3 matrix is 6 and the product of the Eigen values is also 6. If one of the Eigen values is one, find the other two Eigen values.

Sol^n:- let $\lambda_1, \lambda_2, \lambda_3$ be the e.values

$\lambda_1 + \lambda_2 + \lambda_3 = 6$ also $\lambda_1 = 1$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 6$

$$\Rightarrow \left.\begin{array}{l} \lambda_2 + \lambda_3 = 5 \\ \lambda_2 \cdot \lambda_3 = 6 \end{array}\right\} \Rightarrow \lambda_2 = 2, \ \lambda_3 = 3.$$

$\lambda_2 = (5 - \lambda_3)$

$(5 - \lambda_3)\lambda_3 = 6$

$5\lambda_3 - \lambda_3^2 = 6$ $\Rightarrow \lambda_3^2 - 5\lambda_3 + 6 = 0$

$\Rightarrow (\lambda_3 - 2)(\lambda_3 - 3) = 0$

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$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\lambda^3 - 6\lambda^2 + S_2\lambda - 6 = 0$

for $\lambda = 1$, $(1)^3 - 6 + S_2 - 6 = 0 \Rightarrow S_2 = 11$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow \lambda = 1, 2, 3.$

5. If $A = \begin{bmatrix} sin\theta & cosec\theta & 1 \\ \sec\theta & cos\theta & 1 \\ tan\theta & cot\theta & 1 \end{bmatrix}$ then prove that there does not exist a real value of θ for which characteristic roots of A are $-1, 1, 3$

$\lfloor tan\theta \quad cot\theta \quad 1 \rfloor$

characteristic roots of A are $-1, 1, 3$

Soln :- Sum of eigen values = trace of A

$$-1+1+3 = \sin\theta + \cos\theta + 1$$

$$3 = \sin\theta + \cos\theta + 1$$

$\Rightarrow \sin\theta + \cos\theta = 2$

$\Rightarrow$ This is not possible for any real value of θ

$\Rightarrow$ $-1, 1, 3$ can not be e.values for any real value of θ

6. Find the characteristic roots of $A^{30} - 9A^{28}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Soln :- ch. eqn of A is $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)^2 - 4 = 0$$

$$1 - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = -1, 3.$$

matrix	λ_1	λ_2
A	-1	3
A^{28}	$(-1)^{28} = 1$	3^{28}
$-9A^{28}$	$9 \times 1 = 9$	$9 \times 3^{28} = 3^{30}$

If λ is eigen value of A then $f(\lambda)$ is eigen value for $f(A)$

$f(A) = A^{30} - 9A^{28}$

$\therefore f(-1)$ & $f(3)$ are the

	$= 9$	$= 3^{30}$
✓ A^{30}	$(-1)^{30} = 1$	3^{30}
$A^{30} - 9A^{28}$	$1 - 9 = -8$	$3^{30} - 3^{30} = 0$

$\therefore f(-1)$ & $f(3)$ are the e-values for $f(A)$

$f(-1) = (-1)^{30} - 9(-1)^{28} = -8$

$f(3) = 3^{30} - 9(3)^{28} = 0$

$\therefore$ The eigen values of $A^{30} - 9A^{28}$ are -8 & 0

7. Find the Eigen values of $A^2 - 2A + I$ if $A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

Eigen values of A are $1, 2, 3$ (upper triangular matrix)

$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$

$f(A) = A^2 - 2A + I$

Eigen values of $f(A)$ are $f(\lambda_1), f(\lambda_2)$ & $f(\lambda_3)$

$$\begin{cases} f(\lambda_1) = f(1) = (1)^2 - 2(1) + 1 = 1 - 2 + 1 = 0 \\ f(\lambda_2) = f(2) = (2)^2 - 2(2) + 1 = 1 \\ f(\lambda_3) = f(3) = (3)^2 - 2(3) + 1 = 4 \end{cases}$$

$\therefore$ Eigen values of $A^2 - 2A + I$ are $0, 1, 4$

8. Find the Eigen values of $adjA$ if $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

$$A \longrightarrow \quad 1 \quad \Big| \quad 2 \quad \Big| \quad 4 \quad \Big| \quad 6$$

$$\operatorname{adj} A \rightarrow \quad \frac{|A|}{1} \quad \Big| \quad \frac{|A|}{2} \quad \Big| \quad \frac{|A|}{4} \quad \Big| \quad \frac{|A|}{6}$$

$$= 48 \quad \Big| \quad 24 \quad \Big| \quad 12 \quad \Big| \quad 8$$

but $|A|$

$= 1 \times 2 \times 4 \times 6$

$= 48$

9. If A is a square matrix of order 2 with $|A| = 1$ then prove that A and A^{-1} have the same eigen values. Hence verify for $A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

Soln :- Eigen values of A are α and β

$\therefore$ Eigen values of A^{-1} are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

Also $|A| = 1 \Rightarrow \alpha\beta = 1 \Rightarrow \alpha = \frac{1}{\beta}$ & $\beta = \frac{1}{\alpha}$

$\therefore$ Eigen values of A^{-1} are β & α

$\therefore$ A & A^{-1} have same eigen values.

Now $A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ $\qquad |A| = -1 + 2 = 1$

Ch. eqn of A is $\begin{vmatrix} -1-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$(-1-\lambda)(1-\lambda) + 2 = 0$

$-1 + \lambda - \lambda + \lambda^2 + 2 = 0$

$A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ $\qquad \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

Now $A^{-1} = \frac{\operatorname{adj} A}{|A|} = \operatorname{adj} A = \begin{bmatrix} 1 & -2 \end{bmatrix}$

Now $A^{-1} = \frac{adj A}{|A|} = adj A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

ch. eqn of A^{-1} is $\begin{vmatrix} 1-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(-1-\lambda) + 2 = 0$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$\therefore$ verified for the given matrix A.

10. Verify that $X = [2,3,-2,-3]'$ is an eigen vector corresponding to the eigen value $\underline{\lambda = 2}$ of the matrix.

$$A = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$$

Soln :- To check that $\boxed{AX = \lambda X}$ ✓

$$AX = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -4 \\ -6 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix}$$

$$= 2X$$

$\therefore AX = 2X = \lambda X$

$\therefore X$ is the eigen vector for eigen value $\lambda = 2$ for matrix A.