CAYLEY – HAMILTON THEOREM

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STATEMENT: Every square matrix satisfies it's characteristic Equation.

$A \rightarrow$ ch. eqn $\rightarrow \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_0 = 0$.

cayley- Hamilton theorem means that

$$A^n - a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_0 I = 0.$$

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \qquad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

By C-H thm

$$A^3 - 6A^2 + 11A - 6I = 0$$

1. Verify the Cayley-Hamilton theorem for the matrix A and hence, find A^{-1} and A^4 where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Soln: The characteristic equation is $\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

To verify cayley-Hamilton theorem, we have to show that $A^3 - 5A^2 + 9A - I = 0$ —— (1)

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

Sub in LHS of 1

$A^3 - 5A^2 + 9A - I$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Hence Cayley-Hamilton Theorem is verified.

$A^3 - 5A^2 + 9A - I = 0$ —— 1

multiply throughout by A^{-1}

$A^2 - 5A + 9I - A^{-1} = 0$

$\Rightarrow A^{-1} = A^2 - 5A + 9I = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

multiplying 1 by A

$A^4 - 5A^3 + 9A^2 - A = 0$

$A^4 = 5A^3 - 9A^2 + A = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$

2. Find the characteristic equation of the matrix A given below and hence, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

soln: The characteristic eqn of A is $\begin{vmatrix} 2-\lambda & 1 & 1 \end{vmatrix}$

$\ldots$ where $A = \begin{bmatrix} \ldots \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Soln :- The characteristic eqn of A is $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley-Hamilton theorem

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{--- (1)}$$

Now dividing $(\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1)$ by $(\lambda^3 - 5\lambda^2 + 7\lambda - 3)$

$$\begin{array}{r|l} & \lambda^5 + \lambda \\ \hline \lambda^3 - 5\lambda^2 + 7\lambda - 3 & \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 \\ & \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 \\ & - \quad + \quad - \quad + \\ \hline & \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 \\ & \lambda^4 - 5\lambda^3 + 7\lambda^2 - 3\lambda \\ & - \quad + \quad - \quad + \\ \hline & \lambda^2 + \lambda + 1 \end{array}$$

$\therefore$ quotient $= \lambda^5 + \lambda$

remainder $= \lambda^2 + \lambda + 1$.

dividend = (divisor) x (quotient) + Remainder

writing this in terms of A.

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= \underbrace{(A^3 - 5A^2 + 7A - 3I)}(A^5 + A) + (A^2 + A + I)$$

$$= 0 + (A^2 + A + I) \quad \text{(using (1))}$$

$$= A^2 + A + I$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore \text{given expression} = A^2 + A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

3. Apply Cayley-Hamilton theorem to $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and deduce that $A^8 = 625I$

Soln :- Characteristic eqn of A $\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(-1-\lambda) - 4 = 0$$

$$\lambda^2 - 5 = 0$$

By cayley-Hamiton theorem

$$A^2 - 5I = 0 \quad \text{—— (1)}$$

$$\boxed{A^2 = 5I}$$

$$A^2 \cdot A^2 = 5I \cdot 5I$$

$$A^4 = 25I$$

$$A^4 \cdot A^4 = (25I)(25I)$$

$A^8 = 625 I$

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4. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$, hence, find A^{50} ✓

Soln:- The characteristic equation of A is $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix}$

$$\Rightarrow (1-\lambda)\left[\lambda^2 - 1\right] = 0$$

$$\lambda^2 - 1 - \lambda^3 + \lambda = 0$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

$\therefore$ By cayley- Hamilton theorem,

$$A^3 - A^2 - A + I = 0 \quad \text{———} \quad (1)$$

Let $P(n)$: $A^n = A^{n-2} + A^2 - I$, $n \geq 3$

we will prove the result by the method of mathematical Induction.

for $n = 3$,

$P(3)$: $A^3 = A + A^2 - I$ this is true from (1)

Let us assume that $P(n)$ is true for $n = K$

ie $A^K = A^{K-2} + A^2 - I$ ——— (2)

TPT: $P(n)$ is true for $n = K+1$

$$A^{K+1} = A \cdot A^K = A \cdot \left(A^{K-2} + A^2 - I\right) \text{ (using (2))}$$

$$= A^{(k+1)-2} + A^3 - A$$

$$= A^{(k+1)-2} + (A + A^2 - I) - A \quad (\text{using } 1)$$

$$A^{k+1} = A^{(k+1)-2} + A^2 - I$$

$$\{P(n) : A^n = A^{n-2} + A^2 - I\}$$

$\therefore$ $P(n)$ is true for $n = k+1$

$\therefore$ By mathematical induction,

$P(n)$: $A^n = A^{n-2} + A^2 - I$ is true for $n \geq 3$.

To find A^{50}, put $n = 50$ in $P(n)$

$$A^{50} = A^{48} + A^2 - I$$

$$A^{50} = (A^{46} + A^2 - I) + (A^2 - I) \quad (n = 48)$$

$$A^{50} = A^{46} + 2(A^2 - I)$$

$$= A^{44} + A^2 - I + 2(A^2 - I)$$

$$A^{50} = A^{44} + 3(A^2 - I)$$

$$\vdots \qquad \vdots \qquad \vdots$$

continuing like this 24 steps.

$$A^{50} = A^2 + 24(A^2 - I)$$

$$A^{50} = 25A^2 - 24I$$

Now $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\therefore A^{50} = 25 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - 24 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$