SIMILARITY OF MATRICES

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$A, B \rightarrow M$ (non singular)

$M^{-1}AM = B \rightarrow$ similarity

SIMILARITY OF MATRICES

Definition:

(i) If A and B are two square matrices of order n then B is said to be **similar** to A if there exists a non – singular matrix M such that $B = M^{-1}AM$

(ii) A square matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix.

Combining the two definitions we see that A is diagonalizable if there exists a matrix M such that

$M^{-1}AM = D$

where D is a diagonal matrix. In this case M is said to diagonalize A or transform A to diagonal form.

A $\downarrow$ M, D $\downarrow$ $M^{-1}AM = D$

Theorem 1: If A is similar to B and B is similar to C, then A is similar to C. $\rightarrow$ transitive.

Theorem 2: If A and B are similar matrices then $|A| = |B|$

Theorem 3: If A is similar to B, then A^2 is similar to B^2

Corollary: If A is diagonalisable then A^2 is diagonalisable.

Theorem 4: If A and B are two similar matrices then they have the same Eigen values

ALGEBRAIC AND GEOMETRIC MULTIPLICITY OF AN EIGEN VALUES (AM, GM)

Definition:

(i) If λ is an eigen value of the matrix A repeated t times then t is called the algebraic multiplicity of λ.

(ii) If s is the number of linearly Independent Eigen vectors corresponding to the eigen value λ then s is called the geometric multiplicity of λ.

EX-1

$\lambda = 1, 2, 3$

$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \lambda = 1$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \lambda = 2$

$X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \lambda = 3$

E.V.	AM	GM
1	1	1
2	1	1

EX-2

$\lambda = 5, 1, 1$

$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \lambda = 5$

$\left.\begin{array}{l} X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \end{array}\right\} \lambda = 1$

EV	AM	GM
5	1	1

EX-3

$\lambda = 1, 2, 2$

$X_1 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \quad \lambda = 1$

$X_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \quad \lambda = 2$

EV	AM	GM
1	1	1

EX-4

$\lambda = 2, 2, 2$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \lambda = 2$

$\lambda = 2$

$AM = 3$

1	1	1
2	1	1
3	1	1

✓

EV	AM	GM
5	1	1
1	2	2

✓

EV	AM	GM
1	1	1
2	2	1

✗

AM = 3
GM = 1

✗

diagonalisable

Theorem: The necessary and sufficient condition of a square matrix to be similar to a diagonal matrix is that the geometric multiplicity of each of its Eigen values coincides with the algebraic multiplicity.

i. e. We can diagonalise a given square matrix if and only if algebraic multiplicity of each of its Eigen values is equal to the geometric multiplicity. for every eigen value AM = GM $\rightarrow$ diagonalisable

If corresponding to any Eigen value, if algebraic multiplicity is **not equal** to geometric multiplicity then the matrix is **not diagonalizable.**

$M^{-1}AM = D$

Corollary: Every matrix whose Eigen values are distinct is similar to a diagonal matrix.

Theorem: A square nonsingular matrix A whose Eigen value are all distinct can be diagonalised by a similarity transformation $D = M^{-1}AM$ where M is the matrix whose columns are the Eigenvectors of A and D is the diagonal matrix whose diagonal elements are the Eigen values of A.

$\lambda = 1, 2, 3$ $\quad X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$M = \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Notes: **1.** If Eigen values of A are not distinct then it may or may not be possible to diagonalise it.

2. A and D are similar matrices and hence, they have the same Eigen values

3. The process of finding the modal matrix M is called diagonalising the matrix A.

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1. Find the algebraic multiplicity and geometric multiplicity of each Eigen value of the matrix

$$\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

Soln: The characteristic Equation is $\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0.$$

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0.$$

Eigen values = 2, 2, 3

$\therefore$ For $\lambda = 2$, AM = 2

For $\lambda = 3$, AM = 1

For $\lambda = 2$, $[A - \lambda I]X = 0 \Rightarrow [A - 2I]X = 0$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow[R_3 - 3R_1]{R_2 + 2R_1} \begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow[-\frac{1}{5}R_3]{\frac{1}{3}R_2} \begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ Rank = 2

$\therefore$ no. of parameters = 3 − 2
= 1

No. of eigen vectors = 1.

$x_1 + 10x_2 + 5x_3 = 0$

$5x_2 + 2x_3 = 0$

Let $x_3 = 5t \Rightarrow x_2 = -2t$

$x_1 - 20t + 25t = 0 \Rightarrow x_1 = -5t$

$$\therefore X_1 = \begin{bmatrix} -5t \\ -2t \\ 5t \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 5 \end{bmatrix}$$ is an eigen vector for $\lambda = 2$

For $\lambda = 3$, $[A - 3I]X = 0$

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

By crammer's Rule

$x_1 + 3x_2 + 2x_3 = 0 \quad \left(-\frac{1}{2}R_2\right)$

$3x_1 + 5x_2 + 4x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix}}$$

$$\frac{x_1}{2} = \frac{-x_2}{-2} = \frac{x_3}{-4}$$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is an eigen vector for $\lambda = 3$

For $\lambda = 2$, AM $= 2$ GM $= 1$.

For $\lambda = 3$, AM $= 1$ GM $= 1$.

Note :- This matrix is not diagonalisable as AM $\neq$ GM for $\lambda = 2$.

2. Show that the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix

Sol^n :- Ch. eq^n $\longrightarrow$ $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda \longrightarrow 0, 3, 15$

$|A| = 0$.
eigen value $= 0$.
If A is a

$\lambda \longrightarrow 0, 3, 15$

Since, all Eigen values are distinct, the matrix A is diagonalisable.

$= 0.$

If A is a singular matrix then atleast one of the eigen values $= 0$

For $\lambda = 0$, $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

For $\lambda = 3$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 15$, $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

The matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ will be diagonalised to diagonal matrix $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ by transforming matrix $M = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ such that $\underline{\underline{M^{-1}AM = D}}$.

3. Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonisable. Find the diagonal form D and the diagonalising matrix M

Soln: ch. eqn $\longrightarrow \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

Soln :- ch. eqn $\longrightarrow \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

$\lambda \rightarrow -1, -1, \underline{\underline{3}}$

For $\lambda = -1$, $[A - \lambda I]X = 0 \Rightarrow [A + I]X = 0$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ Rank = 1. $\therefore$ no. of parameters $= 3 - 1 = 2$

$\therefore$ No. of L.I. eigen vecters = 2

$$-8x_1 + 4x_2 + 4x_3 = 0$$

let $x_2 = t$, $x_3 = s$

$8x_1 = 4t + 4s \Rightarrow x_1 = \frac{t}{2} + \frac{s}{2}$

$$\therefore X = \begin{bmatrix} \frac{t}{2} + \frac{s}{2} \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ & $X_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ are eigen vecters for $\lambda = -1$

For $\lambda = 3$, $[A - \lambda I]X = 0 \Rightarrow [A - 3I]X = 0$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is eigen vector for $\lambda = 3$ (check)

$\therefore$ For $\lambda = -1$ AM $= 2$ GM $= 2$

For $\lambda = 3$ AM $= 1$ GM $= 1$

$\therefore$ A is diagonalisable as AM = GM for all the eigen values.

The given matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ will be diagonalised to the diagonal form $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transforming matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ such that $M^{-1}AM = D$.

4. Show that the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is not similar to a diagonal matrix (not diagonalisable).

Soln :- eigen values $\rightarrow 2, 2, 1$

(A is a triangular matrix)

For $\lambda = 2$, $[A - 2I]X = 0$

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{R_3 - R_2} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ Rank = 2

$\therefore$ no. of parameters = 3 - 2 = 1

no. of L.I. eigen vectors = 1

$$\left.\begin{array}{r} 3x_2 + 4x_3 = 0 \\ -x_3 = 0 \end{array}\right\} \Rightarrow x_2 = 0, x_3 = 0$$

$\therefore$ $x_1 = t$

$\therefore$ $X_1 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigen vector for $\lambda = 2$

$\therefore$ For $\lambda = 2$, AM = 2
GM = 1.

$\therefore$ AM $\neq$ GM for $\lambda = 2$

$\Rightarrow$ A is not diagonalisable

Ex:- $A = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} = KI \quad (K \in R)$

Eigen values = $\lambda = 50, 50, 50$ (diagonal matrix)

AM for $\lambda = 50$ is 3

For $\lambda = 50$, $[A - 50I] X = 0$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

Rank = 0

no. of parameters = 3 - 0 = 3

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Rank $= 0$

no. of parameters $= 3 - 0 = 3$

let $x_1 = p$, $x_2 = q$, $x_3 = r$

$$X = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = p\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + q\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

are the eigen vector for $\lambda = 50$.

$\therefore$ GM for $\lambda = 50$ is

$\therefore$ AM = GM for $\lambda = 50 \Rightarrow$ A is diagonalisable.

<u>Note</u> :- X_1, X_2, X_3 are Linearly Independent (check).

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5. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix}$, prove that both A and B are not diagonable but AB is diagonable

<u>Soln</u> :- $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Eigen values of A are $\lambda = 1, 1$ (upper triangular)

AM of $\lambda = 1$ is 2

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow x_2 = 0$

$$\begin{bmatrix} \ & \ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ \\ x_2 \end{bmatrix}$$

$\Rightarrow 2x_2 = 0 \Rightarrow x_2 = 0$

Let $x_1 = 1$.

$\therefore X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is eigen vecter corresponding to $\lambda = 1$

$\therefore$ GM of $\lambda = 1$ is 1.

$\therefore$ AM $\neq$ GM for $\lambda = 1$

$\therefore$ matrix A is not diagonalisable

For $B = \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix}$

Eigen values for B, $\lambda = 2, 2$
(lower triangular)

$\therefore$ AM for $\lambda = 2$ is 2

For $\lambda = 2$, $[B - \lambda I]X = 0 \Rightarrow [B - 2I]X = 0$

$$\begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \frac{1}{2}x_1 = 0 \Rightarrow x_1 = 0$$

Let $x_2 = 1$

$\therefore X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is eigen vecter for $\lambda = 2$

GM for $\lambda = 2$ is 1

AM $\neq$ GM for $\lambda = 2$

AM $\neq$ GM for $\lambda = 2$

$\therefore$ B is not diagonalisable.

Let $C = AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1/2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1/2 & 2 \end{bmatrix}$

Charateristic eqn for C is $\begin{vmatrix} 3-\lambda & 4 \\ 1/2 & 2-\lambda \end{vmatrix} = 0$

$$(3-\lambda)(2-\lambda) - 2 = 0$$
$$\lambda^2 - 5\lambda + 4 = 0$$
$$\lambda = 1, 4$$

C = AB has distinct eigen values.

$\therefore$ C is diagonalisable.

For $\lambda = 1$, $[C - \lambda I] X = 0 \Rightarrow [C - I] X = 0$

$\begin{bmatrix} 2 & 4 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow R_2 - \frac{1}{4} R_1 \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$2x_1 + 4x_2 = 0$$

Let $x_2 = t \Rightarrow x_1 = -2t$

$\therefore X_1 = \begin{bmatrix} -2t \\ t \end{bmatrix} \sim \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is eigen vector for $\lambda = 1$.

For $\lambda = 4$, $[C - \lambda I] X = 0 \quad [C - 4I] X = 0$

$X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

6. Find the symmetric matrix $A_{3\times3}$ having the eigen values $\lambda_1 = 0, \lambda_2 = 3$ and $\lambda_3 = 15$, with the corresponding Eigen vectors $X_1 = [1, 2, 2]', X_2 = [-2, -1, 2]'$ and X_3