# EIGEN VALUES & EIGEN VECTORS

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$(-1)^n$

$(-1)^2 \rightarrow \lambda^2 - 2\lambda - 3 = 0$

trace of A

$|A|$

## DEFINITION

Let A be a square matrix of order n

$A - \lambda I$ is a matrix known as characteristic matrix

$|A - \lambda I| \rightarrow$ polynomial in $\lambda$

↳ characteristic polynomial

$|A - \lambda I| = 0 \rightarrow$ Characteristic Equation

The Roots of characteristic Equation are known as the characteristic roots, Latent roots or propervalues or Eigen Values.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 - 4$

$= \lambda^2 - 2\lambda - 3$

$|A - \lambda I| = 0$

$\boxed{\lambda^2 - 2\lambda - 3 = 0}$

$\lambda = -1, 3$

## PROPERTIES OF THE CHARACTERISTIC POLYNOMIAL:

**(1)** The characteristic polynomial $|A - \lambda I\ |$ of a matrix A is an ordinary polynomial in $\lambda$ of degree $n$ where A is a square matrix of order n

**(2)** In characteristic polynomial of A

**(i)** the coefficient of $\lambda^n$ is $(-1)^n$

**(ii)** the coefficient of $\lambda^{n-1}$ is trace of A

(iii) the constant term is $|A|$

**(3)** If A is $3 \times 3$ matrix then the characteristic equation can be expressed as

$|A - \lambda I\ | = (-1)^3\lambda^3 + (-1)^2 S_1 \lambda^2 + (-1) S_2 \lambda + |A| = 0$

Where $S_1 =$ Sum of the diagonal elements of A,

$S_2 =$ Sum of the minors of the diagonal elements of $A$

Ch. eqn can be written as (3x3 matrix)

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$S_1$ = trace of A = sum of diagonal elements of A

$S_2$ = sum of minors of the diagonal elements of A

**PROPERTIES OF THE CHARACTERISTIC ROOTS (EIGEN VALUES):**

**(1)** If A is a square matrix of order n then the degree of the characteristic equation is $n$ and consequently there are exactly n roots (eigenvalues) not necessarily distinct.

**(2)** Sum of all eigenvalues = The sum of the diagonal elements of A (i.e. trace of A)

**(3)** Product of all eigenvalues of $A = |A|$ = constant term in the polynomial.

**CHARACTERISTIC VECTORS OR EIGEN VECTORS:**

Let A be a nxn square matrix.

let $\lambda$ be one of it's Eigen values

Then there exists a non-zero vector X such that

$\boxed{AX = \lambda X}$ → $AX - \lambda X = 0$

$[A - \lambda I]X = 0$ → Homogeneous system of eqns

only trivial → rank = n

non-trivial → rank < n

$|A - \lambda I| = 0$

$\lambda$

This X is the eigen vector corresponding to $\lambda$.

**WORKING RULE TO FIND THE EIGEN VECTORS OF A:**

For the given eigen value $\lambda$,

we take $[A - \lambda I]X = 0$ → Homogeneous system

↓ row trans.

Echelon form

or use crammer's Rule to find the solution

**SOME SOLVED EXAMPLES:**

**1.** Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

soln :- The characteristic Equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)\left[(-3-\lambda)(1-\lambda) - 8\right] + 8\left[4(1-\lambda) + 6\right]$$
$$-2\left[-16 - 3(-3-\lambda)\right] = 0$$

$$\boxed{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0}$$

OR use the formula instead

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$S_1$ = trace of $A$

$= 8 + (-3) + 1 = 6$

$S_2$ = sum of minors of diagonal elements of $A$

= minor of $8$ + minor of $(-3)$ + minor of $(1)$

$$= \begin{vmatrix} -3 & -2 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -8 \\ 4 & -3 \end{vmatrix}$$

$$S_2 = -3 - 8 + 8 + 6 - 24 + 32$$

$\therefore S_2 = 11$

$|A| = 6$

$\therefore$ The characteristic equation is

$$\boxed{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0}$$

The roots are $\boxed{\lambda = 1, 2, 3}$ → eigen values

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

use crammer's Rule

choose any two rows and write the corresponding equations

$$7x_1 - 8x_2 - 2x_3 = 0$$
$$4x_1 - 4x_2 - 2x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\frac{x_1}{8} = \frac{-x_2}{-6} = \frac{x_3}{4}$$

$$\frac{x_1}{4} = \frac{x_2}{3} = \frac{x_3}{2}$$

$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ is eigen vecter correspon to eigen value $\lambda = 1$

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For $\lambda = 2$, $[A - \lambda I] X = 0 \Rightarrow [A - 2I] X = 0$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Taking $1^{st}$ & $2^{nd}$ row

$$6x_1 - 8x_2 - 2x_3 = 0$$
$$4x_1 - 5x_2 - 2x_3 = 0$$

By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{-x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is the eigen vecter corresponding to $\lambda = 2$

For $\lambda = 3$, $[A - \lambda I] X = 0 \Rightarrow [A - 3I] X = 0$

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Choose $1^{st}$ & $2^{nd}$ row

$$5x_1 - 8x_2 - 2x_3 = 0$$
$$4x_1 - 6x_2 - 2x_3 = 0$$

By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-2} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vecter corresponding to $\lambda = 3$

**2.** Find the Eigen values and Eigen vectors of the matrix. Also verify that the Eigen vectors are linearly independent $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Soln:- The characteristic Equation is $\begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0.$$
$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$\therefore$ Eigen values are $\lambda = 1, 2, 3$

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = 0$$

By $\xrightarrow[R_3 - R_1]{R_2 - R_1}$ $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$x_1 - x_2 + x_3 = 0$

$2x_2 - 2x_3 = 0 \Rightarrow x_2 = x_3$

let $x_3 = t \Rightarrow x_2 = t$

$x_1 - t + t = 0 \Rightarrow \boxed{x_1 = 0.}$

$\therefore X_1 = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vecter for $\lambda = 1$.

For $\lambda = 2$, $[A - \lambda I]X = 0 \Rightarrow [A - 2I]X = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \left.\begin{aligned} -x_2 + x_3 &= 0 \\ x_1 - x_3 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 = x_2 = x_3$$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vecter corresponding to $\lambda = 2$

$\lambda = 3$, $[A - \lambda I]X = 0 \Rightarrow [A - 3I]X = 0$

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\xrightarrow[R_3 + R_1]{R_2 + R_1}$ $\begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\xrightarrow[R_3+R_1]{R_2+R_1} \begin{bmatrix} 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad -x_1 - x_2 + x_3 = 0$$

$$-2x_2 = 0 \Rightarrow \boxed{x_2 = 0}$$

$$-x_1 + x_3 = 0 \Rightarrow \boxed{x_1 = x_3}$$

let $x_1 = x_3 = 1$

$\therefore X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 3$.

Now to check $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

are Linearly Independent.

let $k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$

$$k_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\begin{aligned} k_2 + k_3 &= 0 \\ k_1 + k_2 &= 0 \\ k_1 + k_2 + k_3 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$$\xrightarrow{R_3-R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0 \Rightarrow \begin{aligned} k_1 + k_2 &= 0 \\ k_2 + k_3 &= 0 \end{aligned}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} k_1 + \ldots \\ k_2 + k_3 = 0 \\ k_3 = 0 \end{array}$$

$$\Rightarrow k_1 = k_2 = k_3 = 0$$

$\therefore$ The eigen vectors are Linearly Independent.

**3.** Determine the Eigen values and the associated Eigen vectors for the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Soln:- The characteristic equation is $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$

or $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\lambda^3 - \underline{\underline{7}}\lambda^2 + 11\lambda - \underline{\underline{5}} = 0$

$\therefore$ The Eigen values are $\lambda = 5, 1, 1$

For $\lambda = 5$, $[A - \lambda I]X = 0 \Rightarrow [A - 5I]X = 0$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

By crammer's Rule

$-3x_1 + 2x_2 + x_3 = 0$

$x_1 - 2x_2 + x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vecter corresponding to $\lambda = 5$

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Rank $= 1$, $\therefore$ no. of parameters $= 3 - 1 = 2$

$x_1 + 2x_2 + x_3 = 0$

let $x_1 = t$, $x_3 = s \Rightarrow x_2 = -\frac{1}{2}(t+s)$

$$\therefore X = \begin{bmatrix} t \\ -\frac{1}{2}(t+s) \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ & $X_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ are eigen vecters corresponding to $\lambda = 1$.

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**4.** Find the Eigen values and Eigen vectors for $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

Soln: The characteristic Eqn is

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

Sum of eigen values = trace = 5

$\therefore$ roots are $\lambda = 1, 2, 2$

For $\lambda = 1$, $[A - \lambda I]X = 0 \Rightarrow [A - I]X = 0$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} x_1 + 2x_2 + 2x_3 = 0 \\ x_1 + 5x_2 + 3x_3 = 0 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ 5 & 3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix}}$$

$$\frac{x_1}{-4} = \frac{-x_2}{1} = \frac{x_3}{3}$$

$\therefore X_1 = \begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix}$ is an eigen vector for $\lambda = 1$.

For $\lambda = 2$, $[A - \lambda I]X = 0 \Rightarrow [A - 2I]X = 0$

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}$$

$$\xrightarrow[R_3 + R_1]{R_2 - R_1} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ Rank = 2

$\therefore$ No. of parameters = 1.

$$\begin{bmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ No. of parameters $= 1$.

$\therefore$ No. of L.I. Eigenvectors $= 1$.

$$x_1 + 3x_2 + 3x_3 = 0$$
$$2x_2 + x_3 = 0$$

let $x_2 = t \Rightarrow x_3 = -2t$

$x_1 + 3t - 6t = 0 \Rightarrow x_1 = 3t$

$\therefore X_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ is an eigen vector for $\lambda = 2$

**5.** Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Upper triangular
Lower triangular
Diagonal.

Soln, Ch. eqn

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^3 = 0$$

$$\Rightarrow \lambda = 2, 2, 2$$

For $\lambda = 2$, $[A - \lambda I]X = 0 \Rightarrow [A - 2I]X = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ rank $= 2$

$\Rightarrow$ no. of parameters $= 3 - 2 = 1$

$\Rightarrow$ 1 Eigen vector.

$x_2 = 0$
$x_3 = 0$ let $x_1 = t$

$X = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigen vector for $\lambda = 2$.

$X = \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigen vector for $\lambda = 2$.

$\longrightarrow$ Unitary matrix $\rightarrow AA^{\theta} = I$

**6.** Prove that the Eigen values of $\begin{bmatrix} \frac{(1+i)}{2} & \frac{-(1-i)}{2} \\ \frac{(1+i)}{2} & \frac{(1-i)}{2} \end{bmatrix}$ are of unit modulus

Sol$^n$:- Ch. eqn is

$$\begin{vmatrix} \frac{(1+i)}{2} - \lambda & \frac{-(1-i)}{2} \\ \frac{(1+i)}{2} & \frac{(1-i)}{2} - \lambda \end{vmatrix} = 0$$

$$\left[\frac{(1+i)}{2} - \lambda\right]\left[\frac{(1-i)}{2} - \lambda\right] - \left[\left(\frac{1+i}{2}\right)\left(\frac{-(1-i)}{2}\right)\right] = 0$$

$$\left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) - \frac{(1+i)}{2}\lambda - \frac{(1-i)}{2}\lambda + \lambda^2 + \left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) = 0$$

$$\frac{1}{2} - \lambda + \lambda^2 + \frac{1}{2} = 0$$

$$\lambda^2 - \lambda + 1 = 0$$

$\therefore$ Eigen values are $\lambda = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$$\lambda = \frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\lambda_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$|\lambda_1| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$|\lambda_2| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = 1$$

$|\lambda_2| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2} = 1$

$\therefore$ Eigen values are of unit modulus.