NUMERICAL METHODS

Wednesday, December 8, 2021 2:11 PM

Analytical method.

ITERATIVE METHODS

The above methods of solving simultaneous linear equations are called **direct methods**. In all these methods the solutions of the equations are arrived at after a certain fixed amount of computations.

There is another class of methods of solving simultaneous equations called **iterative methods**. In these methods we start with certain assumptions as to the values of the variables . By applying a method of this type we get a better approximation. We repeat (iterate) this procedure as many times as we want till we arrive at a desired accuracy.

JOCOBI'S METHOD:

Consider the following system of equations,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2 \quad \text{...................(1)}$$
$$a_3x + b_3y + c_3z = d_3$$

$a_1 > b_1, c_1$

$b_2 > a_2, c_2$

$c_3 > a_3, b_3$

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \longrightarrow (2)$$

First Iteration Let $x_0 = 0$, $y_0 = 0$, $z_0 = 0$

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0) = \frac{d_1}{a_1}$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0) = \frac{d_2}{b_2}$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0) = \frac{d_3}{c_3}$$

second Iteration :- use x_1, y_1, z_1 to calculate x_2, y_2, z_2 using (2)

$$x_2 = \frac{1}{a_1}(d_1 - b_1y_1 - c_1z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

third Iteration :- use x_2, y_2, z_2 in the RHS of (2) to calculate x_3, y_3, z_3

Repeat this process till you reach the desired accuracy level or the desired number of steps.

SOME SOLVED EXAMPLES:

1. Solve the following equations by Jacobi's method

$4x + y + 3z = 17, x + 5y + z = 14, 2x - y + 8z = 12$

Soln:-

$$\left.\begin{aligned} 4x + y + 3z &= 17 \\ x + 5y + z &= 14 \\ 2x - y + 8z &= 12 \end{aligned}\right\} \quad \text{— (1)}$$

$$\left.\begin{aligned} x &= \frac{1}{4}[17 - y - 3z] \\ y &= \frac{1}{5}[14 - x - z] \\ z &= \frac{1}{8}[12 - 2x + y] \end{aligned}\right\} \quad \text{— (2)}$$

First Iteration :- take $x_0 = 0,\ y_0 = 0,\ z_0 = 0$

$$x_1 = \frac{1}{4}[17 - y_0 - 3z_0] = \frac{17}{4} = 4.25$$

$y_1 = \frac{1}{5}[14 - x_0 - z_0] = \frac{14}{5} = 2.8$

$z_1 = \frac{1}{8}[12 - 2x_0 + y_0] = \frac{12}{8} = 1.5$

Second Iteration :- use x_1, y_1, z_1 in the RHS of (2) to calculate x_2, y_2, z_2

$x_2 = \frac{1}{4}[17 - y_1 - 3z_1] = \frac{1}{4}[17 - 2.8 - 3(1.5)] = 2.425$

$y_2 = \frac{1}{5}[14 - x_1 - z_1] = \frac{1}{5}[14 - 4.25 - 1.5] = 1.65$

$z_2 = \frac{1}{8}[12 - 2x_1 + y_1] = \frac{1}{8}[12 - 2(4.25) + 2.8] = 0.7875$

Third Iteration :- use x_2, y_2, z_2 in RHS of (2) to calculate x_3, y_3, z_3

$x_3 = \frac{1}{4}[17 - y_2 - 3z_2] = \frac{1}{4}[17 - 1.65 - 3(0.7875)] = 3.2469$

$y_3 = \frac{1}{5}[14 - x_2 - z_2] = \frac{1}{5}[14 - 2.425 - 0.7875] = 2.1575$

$z_3 = \frac{1}{5}[12 - 2x_2 + y_2] = \frac{1}{8}[12 - 2(2.425) + 1.65] = 1.1$

Fourth Iteration :- use x_3, y_3, z_3 in RHS in (2) to calculate x_4, y_4, z_4

$x_4 = \frac{1}{4}[17 - 2.1575 - 3(1.1)] = 2.8856$

$y_4 = \frac{1}{5}(14 - 3.2469 - 1.1) = 1.9306$

$z_4 = \frac{1}{8}(12 - 2(3.2469) + 2.1575) = 0.9579$

Fifth Iteration

$x_5 = 3.0488, \quad y_5 = 2.0313, \quad z_5 = 1.0199$

$\therefore x = 3, \; y = 2, \; z = 1$ is the approximate solution

GAUSS – SEIDEL METHOD:

This is a modification of Jacobi's method in which as soon as a new approximation of an unknown is obtained, it is used immediately in the next calculation .

Consider as before the system of equations.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2 \quad \text{……………….(1)}$$
$$a_3x + b_3y + c_3z = d_3$$

$a_1 > b_1, c_1$

$b_2 > a_2, c_2$

$c_3 > a_3, b_3$

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \text{—— (2)}$$

First Iteration :- take $x_0 = 0, \; y_0 = 0, \; z_0 = 0$

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0) = \frac{d_1}{a_1}$$

use x_1, z_0 to calculate y_1

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

use x_1, y_1 to calculate z_1

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

Second Iteration :- use y_1, z_1 to calculate x_2

$$x_2 = \frac{1}{a_1}(d_1 - b_1y_1 - c_1z_1)$$

use x_2, z_1 to calculate y_2

$$y_2 = \frac{1}{b_2}(d_2 - a_2x_2 - c_2z_1)$$

use x_2, y_2 to calculate z_2

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_2 - b_3 y_2)$$

Repeat this process till you reach desired accuracy level.

SOME SOLVED EXAMPLES:

1. Solve the following equations by Gauss – Seidel method

$20x + y - 2z = 17, 3x + 20y - z = -18, \quad 2x - 3y + 20z = 25$

Soln:

$$\left.\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \end{aligned}\right\} \text{— (1)}$$

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \text{— (2)}$$

First Iteration let $x_0 = 0, \; y_0 = 0, \; z_0 = 0$

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = \frac{17}{20} = \boxed{0.85}$$

Use x_1, z_0 to calculate y_1

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = \frac{1}{20}(-18 - 3(0.85)) = \boxed{-1.0275}$$

use x_1, y_1 to calculate z_1

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = \frac{1}{20}[25 - 2(0.85) + 3(-1.0275)]$$
$$= \boxed{1.0109}$$

Second Iteration :- use y_1, z_1 to find x_2

$x_2 = \frac{1}{20}[17 - y_1 + 2z_1] = \frac{1}{20}[17-(-1.0275)+2(1.0109)]$

$= 1.0025$

use x_2, z_1 to find y_2

$y_2 = \frac{1}{20}[-18-3(1.0025)+1.0109] = -0.9998$

use x_2, y_2 to calculate z_2

$z_2 = \frac{1}{20}[25-2(1.0025)+3(-0.9998)] = 0.9998$

Third Iteration

$x_3 = \frac{1}{20}[17-(-0.9998)+2(0.9998)] = 0.9999$

$y_3 = \frac{1}{20}[-18-3(0.9999)+0.9998] = -0.99999$

$z_3 = \frac{1}{20}[25-2(0.9999)+3(-0.99999)] = 1.000$

Hence we get $x = 1$, $y = -1$, $z = 1$

2. Solve the following equations by Gauss – Seidel method

$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$, $0.1x_1 + 7x_2 - 0.3x_3 = -19.3$, $0.3x_1 - 0.2x_2 + 10x_3 = 71.4$