LI & LD VECTORS (Linearly Independent and Linearly Dependent)

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Definition: An ordered set of n elements x_i is called n – dimensional vector or a vector of order n denoted by X.

$$X = [x_1\ x_2\ x_3 \ldots\ldots x_n] \longrightarrow \text{row vector}$$

$\bar{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\mathbb{R}^3$

$(1, -2, 3)$

The elements $x_1, x_2, x_3, \ldots, x_n$ are called components of X.

X is denoted by row matrix or column matrix.

column vector

It is more convenient to denote it as column matrix $X = [x_1\ x_2\ x_3 \ldots \ldots x_n]^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_n \end{bmatrix}$

The vector, all of whose components are zero, is called a zero or null vector and is denoted by 0.

$\mathbb{R}^2$

$\hat{i} + 2\hat{j}$

$(1, 2)$

OPERATIONS ON VECTORS:

Algebra of Vectors: Since n – vector is nothing but a row matrix or column matrix, the algebra of vectors can be developed in the same manner as the algebra of matrices.

1. **Equality of Two Vectors:** Two n – vectors $X = [x_1\ x_2\ x_3 \ldots\ldots x_n]$ and $Y = [y_1\ y_2\ y_3 \ldots\ldots y_n]$ are said to be equal if and only if their corresponding components are equal.
   For example, if $X = [a\ b\ c], Y = [2\ 1\ 4]$ and if $X = Y$, then $a = 2, b = 1\ and\ c = 4$.

2. **Addition of Two Vectors:** Let $X = [x_1\ x_2\ x_3 \ldots x_n]\ and\ Y = [y_1\ y_2\ y_3 \ldots\ldots y_n]$ be two n –vectors, then $X + Y = [x_1 + y_1\ x_2 + y_2\ x_3 + y_3 \ldots .\ x_n + y_n]$
   i.e., the sum of two n – vectors is again n – vectors.

3. **Scalar Multiplication:** If k be any scalar and $X = [x_1\ x_2\ x_3 \ldots x_n]$, then $kX = [kx_1\ kx_2\ kx_3 \ldots .\ kx_n]$ is again an n – vector.
   $X = (3, 1, 2) \rightarrow 2X = (6, 2, 4)$

4. **~~Inner Product~~ of Two Vectors:** Let $X = [x_1\ x_2\ x_3 \ldots x_n]$ and $Y = [y_1\ y_2\ y_3 \ldots\ldots y_n]$ be two n – vectors,
   dot product
   then $XY^T = [x_1\ x_2\ x_3 \ldots x_n]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \ldots \\ y_n \end{bmatrix} = [x_1y_1 + x_2y_2 + x_3y_3 + \cdots + x_ny_n]$
   $XY^T = x_1y_1 + x_2y_2 + x_3y_3 + \cdots + x_ny_n$
   Generally we omit the parentheses for a matrix of order 1 ×1.

5. **Length of a Vector or Norm of a Vector:** Let $X = [x_1\ x_2\ x_3 \ldots x_n]$ be a vector, then the length of the vector X is $\sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots . + x_n^2}$ and it is denoted by $\|X\|$
   $\|X\| \rightarrow$ Norm
   $\bar{a} = \hat{i} + \hat{j} + \hat{k}$
   $|\bar{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

6. **Normal Vector:** A vector whose length ~~(unit norm)~~ is one (unity) is called a ~~Normal Vector~~
   i.e., If $\cancel{x_1^2 + x_2^2 + x_3^2 + \ldots . + x_n^2 = 1}$ then $X = [x_1\ x_2\ x_3 \ldots x_n]$ is a normal vector.

   If a vector is not normal, then it can be converted to a normal vector by dividing each of its components by the length of the vector. i.e., If the vector $Y = [y_1\ y_2\ y_3 \ldots\ldots y_n]$ is not normal, then the vector $\left[\frac{y_1}{d}\ \frac{y_2}{d}\ \frac{y_3}{d} \ldots . \frac{y_n}{d}\right]$ is normal, where $d = \sqrt{y_1^2 + y_2^2 + y_3^2 + \ldots \ldots . + y_n^2} = \|Y\|$

   $\bar{b} = \frac{\bar{a}}{\sqrt{3}}$

   $\bar{b} = \frac{\hat{i}}{} + \frac{\hat{j}}{} + \frac{\hat{k}}{\sqrt{3}}$

   For example, Let $X = [2\ 1\ 3]. d = \sqrt{4 + 1 + 9} = \sqrt{14} \neq 1$

$\left[\frac{y_1}{d}\ \frac{y_2}{d}\ \frac{y_3}{d} \ldots\ldots \frac{y_n}{d}\right]$ is normal, where $d = \sqrt{y_1^2 + y_2^2 + y_3^2 + \ldots\ldots + y_n^2} = \|Y\|$

$\sqrt{3}$

$\bar{b} = \frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}$

For example, Let $X = [2\ 1\ 3]$. $d = \sqrt{4+1+9} = \sqrt{14} \neq 1$

$\therefore X$ is not normal but the vector $\bar{X} = \left[2/\sqrt{14} \quad 1/\sqrt{14} \quad 3/\sqrt{14}\right]$ is normal.

perpendicular

7. **Orthogonal Vector:** A vector X is said to be orthogonal to a vector Y if and only if the inner product of X and Y is zero.

   For Example, Let, $X = [1\ \ -3\ 1]$ and $Y = [1\ 1\ 2]$

$$XY^T = [1 - 3\ 1]\begin{bmatrix}1\\1\\2\end{bmatrix} = 1 - 3 + 2 = 0$$

$\therefore$ X and Y are orthogonal vectors.

8. **Linear Combination:** A vector X which can be expressed in the form $X = k_1X_1 + k_2X_2 + \ldots\ldots + k_nX_n$ is said to be linear combination of the set of vectors $X_1, X_2, X_3, \ldots. X_n$. where $k_1, k_2, k_3 \ldots\ldots, k_n$ are any numbers.

$(1,2,3) = 1.(1,0,0) + 2(0,1,0) + 3(0,0,1)$

$\underline{\underline{X}} = k_1 X_1 + k_2 X_2 + k_3 X_3$

$k_1 = 1$
$k_2 = 2$
$k_3 = 3$

$X_1 = (1,0,0)$
$X_2 = (0,1,0)$
$X_3 = (0,0,1)$

LINEARLY DEPENDENT AND INDEPENDENT SET OF VECTORS:

Definition:

Let $X_1, X_2, X_3, \cdots, X_n$ be a set of n vectors.

Let $k_1X_1 + k_2X_2 + \cdots + k_nX_n = 0$ —— (1)

(i) If all $k_1, k_2, \cdots, k_n$ are found to be zero, then we say that $X_1, X_2, \cdots, X_n$ are linearly Independent.

(ii) If at least one of $k_1, k_2, \cdots, k_n$ is non-zero, then the vectors $X_1, X_2, \cdots, X_n$ are said to be Linearly Dependent.

when the vectors are Linearly Dependent, we can write one of the vectors as a linear combination of the remaining vectors.

$x_1 = (1, 2, 2)$, $x_2 = (0, 1, 2)$, $x_3 = (1, 1, 0)$

$x_1 = x_2 + x_3$ $\boxed{x_1 - x_2 - x_3 = 0}$

$k_1 x_1 + k_2 x_2 + k_3 x_3 = \underline{0}$

$k_1 = 1, \ k_2 = -1, \ k_3 = -1$

x_1, x_2, x_3 are Linearly Dependent

* $(3,0,0)$, $(0,5,0)$, $(1,0,7)$ ✓
Linearly Independent.

Steps to check

Given some vectors $x_1, x_2, \dots, x_n$

1. take the linear combination
$k_1 x_1 + k_2 x_2 + \dots + k_n x_n = 0$
2. This will give a homogeneous system of equations. in $k_1, k_2, \dots, k_n$
3. Solve for $k_1, k_2, \dots, k_n$
4. If we get only trivial solution
$\Rightarrow$ vectors are Linearly Independent
5. If we get non-trivial solution
$\Rightarrow$ vectors are Linearly Dependent.

NOTE: (i) when rank of coefficient matrix (i.e., number of non zero rows in echelon form) is equal to number of variables then system has trivial solution and vectors are independent.

(ii) when rank of coefficient matrix (i.e., number of non zero rows in echelon form) is less than number of variables then system has non-trivial solution and it can be obtain by assigning $n - r$ variables as parameter and vectors are dependent.

SOME SOLVED EXAMPLES:

1. Are the vector $X_1 = [1\ 3\ 4\ 2], X_2 = [3\ -5\ 2\ 6], X_3 = [2\ -1\ 3\ 4]$ linearly dependent? If so, express X_1 as a linear combination of the others.

Soln :- Let $k_1 x_1 + k_2 x_2 + k_3 x_3 = 0$ —— 1

Soln :- Let $k_1 x_1 + k_2 x_2 + k_3 x_3 = 0$

$$k_1[1 \;\; 3 \;\; 4 \;\; 2] + k_2[3 \;\; -5 \;\; 2 \;\; 6] + k_3[2 \;\; -1 \;\; 3 \;\; 4] = 0$$

$$\rightarrow [k_1 \;\; 3k_1 \;\; 4k_1 \;\; 2k_1] + [3k_2 \;\; -5k_2 \;\; 2k_2 \;\; 6k_2] + [2k_3 \;\; -k_3 \;\; 3k_3 \;\; 4k_3] = 0$$

$$\rightarrow [k_1+3k_2+2k_3 \quad 3k_1-5k_2-k_3 \quad 4k_1+2k_2+3k_3 \quad 2k_1+6k_2+4k_3] = 0$$

$$\Rightarrow \left.\begin{aligned} k_1+3k_2+2k_3 &= 0 \\ 3k_1-5k_2-k_3 &= 0 \\ 4k_1+2k_2+3k_3 &= 0 \\ 2k_1+6k_2+4k_3 &= 0 \end{aligned}\right\} \text{Homogeneous system of equations}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 6 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 3R_1, \; R_3 - 4R_1, \; R_4 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{1}{7}R_2, \; \frac{1}{5}R_3$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\text{Rank}(A) = 2 < 3$

$\therefore$ There are infinitely many non trivial Solutions.

$\therefore$ The given vectors are Linearly Dependent.

$$k_1 + 3k_2 + 2k_3 = 0$$
$$-2k_2 - k_3 = 0$$

Let $k_2 = t$

$\Rightarrow k_3 = -2t$

$\Rightarrow k_1 + 3t - 4t = 0 \Rightarrow k_1 = t$

Sub. in ①

$$tX_1 + tX_2 - 2tX_3 = 0$$

$$\boxed{X_1 + X_2 - 2X_3 = 0}$$

$\Rightarrow X_1 = -X_2 + 2X_3$

$\Rightarrow \boxed{X_1 = -X_2 + 2X_3}$

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2. Show that the vectors X_1, X_2, X_3 are linearly independent and vector X_4 depends upon them, where $X_1 = [1\ 2\ 4], X_2 = [2\ -1\ 3], X_3 = [0\ 1\ 2], X_4 = [-3\ 7\ 2]$

Soln:- consider $k_1X_1 + k_2X_2 + k_3X_3 = 0$ — ①

$k_1[1\ 2\ 4] + k_2[2\ -1\ 3] + k_3[0\ 1\ 2] = 0$

$k_1 + 2k_2 + 0k_3 = 0$
$2k_1 - k_2 + k_3 = 0$
$4k_1 + 3k_2 + 2k_3 = 0$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 - 2R_1$, $R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & -5 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank $(A) = 3$ $\Rightarrow$ There is only trivial soln

$k_1 + 2k_2 = 0$, $-5k_2 + k_3 = 0$, $k_3 = 0$

$\Rightarrow$ $k_1 = 0$, $k_2 = 0$, $k_3 = 0$

$\therefore$ X_1, X_2, X_3 are Linearly Independent.

Now consider

$$k_1 X_1 + k_2 X_2 + k_3 X_3 + k_4 X_4 = 0$$

$$k_1[1 \; 2 \; 4] + k_2[2 \; -1 \; 3] + k_3[0 \; 1 \; 2] + k_4[-3 \; 7 \; 2] = 0$$

$k_1 + 2k_2 + 0k_3 - 3k_4 = 0$
$2k_1 - k_2 + k_3 + 7k_4 = 0$
$4k_1 + 3k_2 + 2k_3 + 2k_4 = 0$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = 0$$

Applying $R_2 - 2R_1$, $R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = 0$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = 0$$

Rank $(A) = 3 < 4$

$\therefore$ There are infinitely many non trivial solutions

$\therefore$ X_1, X_2, X_3, X_4 are Linearly Dependent.

writing the equations

$$k_1 + 2k_2 + 0k_3 - 3k_4 = 0$$
$$-5k_2 + k_3 + 13k_4 = 0$$
$$k_3 + k_4 = 0$$

let $k_4 = t \Rightarrow k_3 = -t$

$-5k_2 + (-t) + 13(t) = 0 \Rightarrow k_2 = \frac{12}{5}t$

$k_1 + \frac{24}{5}t - 3t = 0 \quad \Rightarrow k_1 = \frac{-9}{5}t$

$k_1 X_1 + k_2 X_2 + k_3 X_3 + k_4 X_4 = 0$

$$-\frac{9}{5}t\,x_1 + \frac{12}{5}t\,x_2 - t\,x_3 + t\,x_4 = 0$$

$$-\frac{9}{5}x_1 + \frac{12}{5}x_2 - x_3 + x_4 = 0$$

$$\Rightarrow \boxed{x_4 = \frac{9}{5}x_1 - \frac{12}{5}x_2 + x_3}$$

$\therefore$ x_4 depends on x_1, x_2, x_3

H.W

3. Examine whether the vectors $X_1 = [1\ 1\ -1], X_2 = [2\ -3\ 5], X_3 = [-2\ 1\ 4]$ are linearly independent.

Soln:- $k_1X_1 + k_2X_2 + k_3X_3 = 0$

$$\begin{bmatrix} 1 & 2 & -2 \\ 1 & -3 & 1 \\ -1 & 5 & 4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$\downarrow$ After row transformations $R_2 - R_1,\ R_3 + R_1$

$$k_1 = k_2 = k_3 = 0$$

$\Rightarrow$ The given vectors are linearly Independent.

4. Show that the following set of vectors are mutually orthogonal vectors
$X_1 = [2\quad 1\quad 2], X_2 = [-2\quad 2\quad 1], X_3 = [1\quad 2\quad -2]$

Soln: $X_1X_2^t = [2\ \ 1\ \ 2]\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = -4+2+2 = 0$

$$X_1X_3^t = [2\ \ 1\ \ 2]\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = 2+2-4 = 0$$

$$X_2X_3^t = [-2\ \ 2\ \ 1]\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = -2+4-2 = 0$$

$$X_2 X_3^t = \begin{bmatrix} -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = -2+4-2=0$$

$\therefore$ X_1, X_2, X_3 are mutually orthogonal.

5. Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ Discuss and find the relation of linear dependence amongst its row vectors.

Soln :- $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

$R_2 - R_1$, $R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix}$$

$R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank}(A) = 2$$

Let $X_1 = \begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 & -1 & 2 & -1 \end{bmatrix}$

$X_3 = \begin{bmatrix} 3 & 1 & 0 & 1 \end{bmatrix}$

$k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$

$k_1 + k_2 + 3k_3 = 0$

$k_1 - k_2 + k_3 = 0$

$-k_1 + 2k_2 + 0k_3 = 0$

$k_1 - k_2 + k_3 = 0$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_2 - R_1, \quad R_3 + R_1, \quad R_4 - R_1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 3 & 3 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$-\frac{1}{2}R_2, \quad \frac{1}{3}R_3, \quad \frac{1}{2}R_4$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_3 + R_2, \quad R_4 - R_2$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$\text{Rank}(A) = 2 < 3$

There are non trivial solutions

$k_1 + k_2 + 3k_3 = 0$

$-k_2 - k_3 = 0$

$$\text{let } k_3 = t \Rightarrow k_2 = -t$$

$$k_1 - t + 3t = 0 \Rightarrow k_1 = -2t$$

$$\therefore k_1 x_1 + k_2 x_2 + k_3 x_3 = 0$$

$$-2t x_1 - t x_2 + t x_3 = 0$$

$$\boxed{-2x_1 - x_2 + x_3 = 0}$$

6. Show that the rows of the matrix $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$ are linearly dependent and express any row as a linear combination of other rows.

Soln :-

$$x_1 = [1 \quad 0 \quad -5 \quad 6] \qquad x_2 = [3 \quad -2 \quad 1 \quad 2]$$

$$x_3 = [5 \quad -2 \quad -9 \quad 14] \qquad x_4 = [4 \quad -2 \quad -4 \quad 8]$$

$$\text{let } k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 = 0$$

$\Downarrow$ write the equations

$$\begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & -2 & -2 & -2 \\ -5 & 1 & -9 & -4 \\ 6 & 2 & 14 & 8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = 0$$

$R_3 + 5R_1, \quad R_4 - 6R_1$

$$\begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & -2 & -2 & -2 \\ 0 & 16 & 16 & 16 \\ 0 & -16 & -16 & -16 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = 0$$

$R_3 + 8R_2, \quad R_4 - 8R_2$

$$\begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_2 \\ k_3 \\ k_4 \end{bmatrix} = 0$$

$$k_1 + 3k_2 + 5k_3 + 4k_4 = 0$$

$$k_2 + k_3 + k_4 = 0$$

no. of parameters $= 4 - 2 = 2$

Let $k_4 = t,\ k_3 = s \Rightarrow k_2 = -s - t$

$$k_1 = -2s - t$$

$$\therefore\ k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 = 0$$

$$(-2s - t)x_1 + (-s - t)x_2 + s x_3 + t x_4 = 0.$$

$$\boxed{s x_3 = (2s + t)x_1 + (s + t)x_2 - t x_4}$$