HOMOGENEOUS SYSTEM OF EQUATIONS

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A SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS:

(a) Solution of $AX = 0$

The null column matrix $(X=0)$ is always a solution of $AX=0$. This solution is called the trivial solution or the zero solution.

Any other solution $X \neq 0$ is called a non-trivial solution or a non-zero solution

(b) Important Results

$AX=0$, let r be the rank of A.

(i) If $r=n$ then there is only trivial sol^n

(ii) If $r<n$ then the system has non-trivial sol^n.
The no. of independent sol^n = no. of parameters $= n-r$

(c) Working Rule to solve Homogeneous Equations

1. Write the given system in matrix form $AX=0$
2. Reduce A to echelon form by applying Row transformations only
3. (a) If rank $A = n \Rightarrow$ only trivial sol^n
   (b) If rank $A < n \Rightarrow$ non trivial sol^n.

SOME SOLVED EXAMPLES:

1. Find the solution of the system given by

$$x_1 - x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 0$$

Soln :-

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Apply $R_2 - R_1$, $R_3 - 2R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Apply $R_3 - R_2$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is echelon form

$\therefore \rho(A) = 3 =$ no. of unknowns

$\therefore$ This system has <u>only</u> trivial soln

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 3x_2 &= 0 \\ x_3 &= 0 \end{aligned}\right\} \Rightarrow x_1 = x_2 = x_3 = 0$$

$$\left.\begin{array}{ll} 3x_2 & = 0 \\ x_3 & = 0 \end{array}\right\} \Rightarrow \ldots$$

The solution is $(x_1, x_2, x_3) = (0, 0, 0)$

2. Solve the following system of linear equations $\begin{array}{r} x_1 + 2x_2 + 4x_3 + x_4 = 0 \\ 2x_1 + x_2 + 5x_3 + 8x_4 = 0 \\ x_1 + 4x_2 + 6x_3 - 3x_4 = 0 \end{array}$

Soln :-
$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 1 & 5 & 8 \\ 1 & 4 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 1 & 5 & 8 \\ 1 & 4 & 6 & -3 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -3 & -3 & 6 \\ 0 & 2 & 2 & -4 \end{bmatrix} \xrightarrow[\frac{1}{2}R_3]{\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -1 & -1 & 2 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form

$\therefore \rho(A) = 2 < 4$ (no. of unknowns)

$\therefore$ There are infinitely many non-trivial solutions

number of parameters $= 4 - 2 = 2$

writing the reduced equations

$$x_1 + 2x_2 + 4x_3 + x_4 = 0 \quad \text{— (i)}$$

$$-x_2 - x_3 + 2x_4 = 0 \quad \text{— (ii)}$$

Let $\boxed{x_2 = t}$, $\boxed{x_3 = s}$

(ii) $\Rightarrow$ $-t - s + 2x_4 = 0$ $\Rightarrow$ $\boxed{x_4 = \dfrac{t+s}{2}}$

(i) $\Rightarrow$ $x_1 + 2t + 4s + \frac{t}{2} + \frac{s}{2} = 0$

$$\Rightarrow \boxed{x_1 = -\frac{5t}{2} - \frac{9}{2}s}$$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{5t}{2} - \frac{9}{2}s \\ t \\ s \\ \frac{t}{2} + \frac{s}{2} \end{bmatrix}$ is the infinitely many non trivial soln as 't' and 's' vary

3. Determine the values of λ for which the following system of equations possess a non – trivial solution and obtain these solutions for each value of λ.

$$3x_1 + x_2 - \lambda x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$$

H.W

4. Solve **(i)** $\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 0 \\ x_1 - x_2 + 2x_3 - x_4 &= 0 \\ 3x_1 + x_2 + x_4 &= 0 \end{aligned}$ **(ii)** $\begin{aligned} 3x_1 + 4x_2 - x_3 - 9x_4 &= 0 \\ 2x_1 + 3x_2 + 2x_3 - 3x_4 &= 0 \\ 2x_1 + x_2 - 14x_3 - 12x_4 &= 0 \\ x_1 + 3x_2 + 13x_3 + 3x_4 &= 0 \end{aligned}$

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5. Discuss for all values of k, the following system of equations possesses trivial and non – trivial solutions

$$2x + 3ky + (3k+4)z = 0$$
$$x + (k+4)y + (4k+2)z = 0$$
$$x + 2(k+1)y + (3k+4)z = 0$$

Soln:- Writing the given system in matrix form

$$\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This system has only trivial solution if

$$\text{Rank}(A) = 3$$

$$\Rightarrow |A| \neq 0$$

Applying R_{12}, we have

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$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 - R_1$

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— } (1)$$

$|A| = 1(\text{cofactor of } 1)$ (expanding over 1^{st} column)

$$= 1(-1)^{1+1} \begin{vmatrix} k-8 & -5k \\ k-2 & -k+2 \end{vmatrix}$$

$|A| = 4k^2 - 16$

Now $|A| = 0 \Rightarrow 4k^2 - 16 = 0 \Rightarrow k^2 = 4 \Rightarrow k = \pm 2$

Case-I :- when $k \neq \pm 2$, $|A| \neq 0$

$\therefore$ The given system will have only trivial solution.

$x = y = z = 0.$

Case-II :- when $k = 2$ then (1) gives

$$\begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2 < 3$

$\Rightarrow$ Infinitely many non trivial solutions

no. of parameters $= n-r = 3-2 = 1$.

writing the reduced equations

$$x + 6y + 10z = 0 \quad \text{— (i)}$$

$$-6y - 10z = 0 \quad \text{— (ii)}$$

Let $\boxed{z = t}$ (t is parameter)

(ii) $\Rightarrow$ $6y = -10t$ $\Rightarrow$ $\boxed{y = \frac{-10t}{6}}$

(i) $\Rightarrow$ $x - 10t + 10t = 0$ $\Rightarrow$ $\boxed{x = 0}$

when $K = 2$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{10t}{6} \\ t \end{bmatrix}$ is the non trivial soln

$t \to$ parameter

(ii) when $K = -2$, (1) gives

$$\begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{R_2}{10}, \frac{R_3}{4}$ $\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 - R_2$ $\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\text{Rank}(A) = 2 < 3$

$\therefore$ Infinitely many non trivial solutions

no of parameters $= 3-2=1$.

writing the reduced equations

$$x+2y-6z=0 \quad \text{—(i)}$$
$$-y+z=0 \quad \text{—(ii)}$$

Let $\boxed{z=s}$ ($s \rightarrow$ parameter)

(ii) $\Rightarrow$ $\boxed{y=s}$

(i) $\Rightarrow$ $x+2s-6s=0 \Rightarrow \boxed{x=4s}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4s \\ s \\ s \end{bmatrix}$$ are the non trivial solution 's' is the parameter.