RANK OF A MATRIX

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ELEMENTARY TRANFORMATIONS:

Following operations on a matrix are called elementary transformations.

1. Interchanging any two rows or any two columns

   $R_i \leftrightarrow R_j$ $\qquad$ $C_i \leftrightarrow C_j$

   R_{ij} $\qquad$ C_{ij}

2. Multiplication of any row/column by a nonzero constant

   $kR_i \rightarrow$ multiplying i^{th} row by k

   $kC_j \rightarrow$ multiplying j^{th} column by k

3. Addition of i^{th} row/column with some multiple of j^{th} row/column

   $R_i \rightarrow R_i + kR_j$ / $C_i \rightarrow C_i + kC_j$

Equivalent Matrices: Two matrices A and B are said to be equivalent if the matrix B is obtained by performing elementary transformations on the matrix A.

The symbol ~ is used for equivalence i.e., $A \sim B$(A is equivalent to B).

$$A \xrightarrow{R} \xrightarrow{C} \xrightarrow{R} B$$

$$A \sim B$$

$$^4C_1 = 4$$

Minor of a Matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 9 \\ -1 & 0 & 5 & 0 \\ 6 & 7 & -3 & -1 \end{bmatrix}$$

If I want a minor of order k, choose any k rows and any k columns from A. (a k×k submatrix)

find determinant of that submatrix

$|6| = 6.$ $\qquad \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2$

The value of this determinant is called a minor of order k.

find minor of order 2

$$^4C_2 \times {}^3C_2$$

$$6 \times 3 = 18$$

no of minors

RANK OF A MATRIX:

Definition: A number 'r' is said to be the rank of matrix A, if it possesses the following properties:

(i) There exists at least one sub – matrix of A of order r whose determinant is non – zero

(ii) Every sub – matrix of A whose determinant with order (r + 1), if it exists, should be zero.

In short, the rank of matrix is the order of any highest order non – vanishing minor.
The rank 'r' of a matrix A is denoted by $\rho(A)$.

A → non zero matrix

A +ve integer r is said to be rank of A if

(i) there exist atleast one minor of order r which is non zero.

(ii) All the minors of order greater than r are zero.

A 5×5 → 5

A → rank - 3 → there is a nonzero minor of order 3

→ all the minors of order 4, 5, 6, ... are zero

If $|A| \neq 0$.

$A \to \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$ 4×3

Note: **(i)** If A is a square matrix of order n, then $1 \leq \rho(A) \leq n$.

(ii) If A is a matrix of order $m \times n$, then $1 \leq \rho(A) \leq \min(m, n)$

(iii) The rank of a null matrix is always zero.

(iv) Rank of a non – singular matrix is always equal to its order.

(v) Rank of a matrix is always unique.

$|A| \neq 0$

$\rho(A)$ rank (A)

$A_{3\times3}$ $1 \leq \rho(A) \leq 3$

Example: Determine the ranks of the following matrices.

(a) $Let\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

Soln:- Since A is square matrix

$|A| = 2 \neq 0$

∴ A is a non singular matrix

∴ rank of A = order of A = 3

(b) $Let\ A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$

Soln $|A| = 0$

∴ rank (A) $\neq 3$

(1st row 2nd row
1st coln 2nd coln)

consider $|1 \ -2|$

consider $\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 4-4=0$ (1st col

Take $\begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} = 2-12$ (2nd col 3rd col
$= -10 \neq 0$ 1st row 2nd row)

$\therefore$ atleast one minor of order 2 is nonzero

Hence $\rho(A)=2$

(c) $Let\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix}$

Soln $|A| = 0$ $\therefore \rho(A) \neq 3$

Any minor of order 2 is also zero

$\therefore \rho(A) \neq 2$

$\therefore \rho(A) = 1.$

observe that, all the rows are identical, so when all the rows of a given matrix are identical then the rank is always 1.

(d) $Let\ A = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 1 & -1 & 0 & 3 \\ 3 & 5 & 1 & 6 \end{bmatrix}_{3\times4}$

A is of order 3×4

$1 \leq \rho(A) \leq 3$

consider 3×3 minor $\begin{vmatrix} 2 & 4 & 3 \\ 1 & -1 & 0 \\ 3 & 5 & 1 \end{vmatrix} = 18 \neq 0$

$\therefore \rho(A) = 3.$

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$A \xrightarrow[\text{col}]{\text{Row}} B$

Note: (i) The elementary transformations of a matrix do not alter the rank of a matrix.
i.e., If $A{\sim}B$, then A and B have same rank.

(ii) Similarly, rank of A = rank of (kA), where k is any scalar.

(iii) If $A_{n\times n}$ is non – singular i.e., $|A| \neq 0$ then rank of $A = n$ and rank of $A^2 = n$
Since $|A^2| = |A.A| = |A|.|A| \neq 0$

(iv) $\rho(A) = \rho(A^T)$

(v) The rank of the product of two matrices cannot exceed the rank of either matrix.

(vi) If r is the rank of the matrix A then the rank of A^n is less than or equal to r.

SOME SOLVED EXAMPLES:

1. Find the ranks of the following matrices

(i) $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & 5 & 6 & 11 \\ 3 & 7 & 10 & 17 \\ 4 & 8 & 12 & 20 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 7 & 8 & 9 & 10 & 11 \\ 22 & 23 & 24 & 25 & 26 \\ 51 & 52 & 53 & 54 & 55 \\ 10 & 11 & 12 & 13 & 14 \end{bmatrix}$

Soln:- (i) $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix} \xrightarrow{C_3 - 3C_2} \begin{bmatrix} 6 & 1 & 0 & 8 \\ 4 & 2 & 0 & -1 \\ 10 & 3 & 0 & 7 \\ 16 & 4 & 0 & 15 \end{bmatrix}$

$R_4 - (R_1 + R_3)$

$\rightarrow \begin{bmatrix} 6 & 1 & 0 & 8 \\ 4 & 2 & 0 & -1 \\ 10 & 3 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - (R_1 + R_2)} \begin{bmatrix} 6 & 1 & 0 & 8 \\ 4 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank can not be 4 or 3

consider the minor of order 2

$$\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 12 - 4 = 8 \neq 0$$

$\therefore$ Rank = 2.

(iv) $\begin{bmatrix} 7 & 8 & 9 & 10 & 11 \\ 22 & 23 & 24 & 25 & 26 \\ 51 & 52 & 53 & 54 & 55 \\ 10 & 11 & 12 & 13 & 14 \end{bmatrix}$

$1 \leq r \leq 4$

$R_2 - R_1, \; R_3 - R_1, \; R_4 - R_1$

$\rightarrow \begin{bmatrix} 7 & 8 & 9 & 10 & 11 \\ 15 & 15 & 15 & 15 & 15 \\ 44 & 44 & 44 & 44 & 44 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$

$\frac{1}{15}R_2, \; \frac{1}{44}R_3, \; \frac{1}{3}R_4$

$\rightarrow \begin{bmatrix} 7 & 8 & 9 & 10 & 11 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 7 & 8 & 9 & 10 & 11 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$R_3 - R_2, \; R_4 - R_2$

$$\rightarrow \begin{bmatrix} 7 & 8 & 9 & 10 & 11 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank can not be 3 or 4

Consider a minor of order 2

$$\begin{vmatrix} 7 & 8 \\ 1 & 1 \end{vmatrix} = 7 - 8 = -1 \neq 0$$

$\therefore$ Rank $= 2$

NORMAL FORM OF A MATRIX:

Definition: By performing elementary row and column transformations, every non – zero matrix can be reduced to one of the four forms, called the normal form of A:

(i) $[I_r]$ **(ii)** $[I_r \; 0]$ **(iii)** $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ **(iv)** $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$A \xrightarrow[\text{col}^n]{\text{Row}} [I_r] \rightarrow$ Identity

$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ $[I_r, 0]$ $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

Note: Rank of A = Rank of the normal form of $A = r$.

Method to Reduce a Given Matrix to its Normal Form by Applying Elementary Transformations:

Step 1: Reduce the first diagonal element a_{11}, which is called a leading element (or a pivot), to 1 by applying any (row or column) transformation

Step 2: Apply row – transformation to reduce all other elements in first column to zero.

Step: 3: Apply column – transformation to reduce all other elements in first row to zero.

Step 4: Reduce the second diagonal element a_{22}, which is then called the leading element, to 1 by applying any (row or column) transformation without disturbing the elements of the first row and first column.

Step 5: Applying row – transformation clear off all other non – zero elements of the second column and reduce them to zero without disturbing the first row.

Step 6: Applying column – transformation clear off all other non – zero elements of the second row and reduce them to zero without disturbing the first column.

$$A \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Continuing the above procedure with the successive rows and columns, we can reduce a given matrix to its normal form.

Note: Application of elementary transformation on any matrix A may differ but rank of A is unique.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

① make $a_{11} \rightarrow 1$

$$A = \begin{bmatrix} 4 & 3 & 0 & -2 \\ 3 & 4 & -1 & -3 \\ 7 & 7 & -1 & -5 \end{bmatrix}$$

$\lfloor a_{m1} \; a_{m2} \; \cdots \rfloor$

(1) make $a_{11} \to 1$

(2) use this 1 to make every element in 1st row and 1st coln as zero.

After this, the matrix changes to

$$\sim \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & B & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

(3) Repeat the process for submatrix B

$\lfloor 7 \; 7 \; -1 \; -5 \rfloor$

(1) $R_1 - R_2$

$$A \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 3 & 4 & -1 & -3 \\ 7 & 7 & -1 & -5 \end{bmatrix}$$

(2) $R_2 - 3R_1, \; R_3 - 7R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 7 & -4 & -6 \\ 0 & 14 & -8 & -12 \end{bmatrix}$$

(3) $C_2 + C_1, \; C_3 - C_1, \; C_4 - C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & -4 & -6 \\ 0 & 14 & -8 & -12 \end{bmatrix}$$

(4) $C_2 + C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & -6 \\ 0 & 2 & -8 & -12 \end{bmatrix}$$

(5) $R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(6) $C_3 + 4C_2, \; C_4 + 6C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

SOME SOLVED EXAMPLES:

1. Reduce the following matrices to their normal form and hence obtain their ranks.

✓**(i)** $\begin{bmatrix} 4 & 3 & 0 & -2 \\ 3 & 4 & -1 & -3 \\ 7 & 7 & -1 & -5 \end{bmatrix}$ H.W. **(ii)** $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 16 & 4 & 12 & 15 \\ 5 & 3 & 3 & 4 \\ 4 & 2 & 6 & -1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & -1 & -2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

(iii) $$A = \begin{bmatrix} 1 & -1 & -2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 5 & 8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$C_2 + C_1, \quad C_3 + 2C_1, \quad C_4 + 3C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 5 & 8 & 14 \end{bmatrix}$$

$R_3 - 3R_2, \quad R_4 - 5R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 8 & 4 \end{bmatrix} \xrightarrow{C_4 - 2C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 8 & 4 \end{bmatrix}$$

$R_4 - 8R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ \ldots \end{bmatrix} \quad C_4 + 2C_3 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \ldots \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 20 \end{bmatrix} \xrightarrow{C_4+2C_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix}$$

$\frac{1}{20} R_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim I_4 \quad \therefore \rho(A) = 4$$

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2. If A and B are as given below, find the rank of A by reducing it to the normal form. Find 3A − B, hence or otherwise, show that $3A^2 - AB = 2A$ also find the rank of $3A^2 - AB$.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 6 & 3 & 5 \\ 2 & 4 & 2 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 4 & 3 & 3 \\ 6 & 18 & 7 & 15 \\ 6 & 12 & 6 & 10 \end{bmatrix}$$

Soln:- $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 6 & 3 & 5 \\ 2 & 4 & 2 & 4 \end{bmatrix} \xrightarrow[R_4-2R_1]{R_3-2R_1} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow[C_4-2C_1]{C_2-2C_1,\; C_3-C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[C_4-C_2]{C_3-C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ This is the normal form of A.

$\therefore$ Rank $(A) = 2$

$\therefore$ Rank $(A) = 2$

Now $3A - B = 3\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 6 & 3 & 5 \\ 2 & 4 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 4 & 3 & 3 \\ 6 & 18 & 7 & 15 \\ 6 & 12 & 6 & 10 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 2I$$

Now $3A^2 - AB = A(3A - B) = A(2I) = 2A.$

$\text{Rank}(3A^2 - AB) = \text{Rank}(2A)$

$= \text{Rank}(A)$

$= 2$

3. Find a, b, c if A is orthogonal matrix, where $A = \frac{1}{3}\begin{bmatrix} a & b & c \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ Hence find the inverse of A and ranks of A^2 and 3A.

Soln:- Since A is orthogonal,

$AA^t = I$

$$\frac{1}{3}\begin{bmatrix} a & b & c \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} a & -2 & 1 \\ b & 1 & -2 \\ c & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This will give

$a^2 + b^2 + c^2 = 9$

$-2a + b + 2c = 0$

$a - 2b + 2c = 0$

Solving these equations, we get

$(a, b, c) = (2, 2, 1)$ or $(a, b, c) = (-2, -2, -1)$

$$\therefore A^{-1} = A^t = \frac{1}{3}\begin{bmatrix} a & -2 & 1 \\ b & 1 & -2 \\ c & 2 & 2 \end{bmatrix}$$

$$\therefore \; A = A' = \frac{1}{3}\begin{bmatrix} & & \\ b & 1 & -2 \\ c & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix} \text{ or } \frac{1}{3}\begin{bmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ -1 & 2 & 2 \end{bmatrix}$$

A is orthogonal matrix

$\Rightarrow |A| \neq 0$

$\Rightarrow$ Rank $(A) = 3$

Also $|A^2| = |A| \cdot |A| \neq 0$

$\therefore$ Rank $(A^2) = 3$

Rank of $3A$ = Rank of $A = 3$.

4. Find the values of P for which the following matrix A will have **(i)** rank 1, **(ii)** rank 2, **(iii)** rank 3, where A is $\begin{bmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{bmatrix}$

Soln:- Rank(A) will be 3 if $|A| \neq 0$

$$|A| = \begin{vmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{vmatrix} = 3(9-p^2) - p(3p-p^2) + p(p^2-3p)$$

$$= 3(3-p)(3+p) - p^2(3-p) + p^2(p-3)$$

$$= (3-p)\left[3(3+p) - p^2 - p^2\right]$$

$$= (3-p)\left[9+3p-2p^2\right]$$

$$|A| = (3-p)^2(3+2p)$$

Now $|A| = 0$

$\Rightarrow (3-p)^2(3+2p) = 0 \Rightarrow p = 3$ or $p = \frac{-3}{2}$

$\therefore$ Rank$(A) = 3$ if $p \neq 3$ and $p \neq \frac{-3}{2}$

when $p = 3$

when $p=3$

$$A=\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

All minors of order 2 are zero

$\therefore \rho(A)=1.$

when $p=-3/2$

$$A=\begin{bmatrix} 3 & -3/2 & -3/2 \\ -3/2 & 3 & -3/2 \\ -3/2 & -3/2 & 3 \end{bmatrix}$$

consider a minor of order 2

$$\begin{vmatrix} 3 & -3/2 \\ -3/2 & 3 \end{vmatrix} = 9-\frac{9}{4}=\frac{27}{4}\neq 0$$

$\therefore \rho(A)=2$

$\therefore$ (i) $\rho(A)=3$ if $p\neq 3$ and $p\neq \frac{-3}{2}$

(ii) $\rho(A)=2$ if $p=-3/2$

(iii) $\rho(A)=1$ if $p=3$

5. If x is real, prove that rank of A is 3, where $A=\begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Soln :- Rank of A is 3 when $|A|\neq 0$

we have $|A|=\begin{vmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 1 & 1 & 1 \end{vmatrix} = x^2-x+1$

If $|A|=0$ then $x^2-x+1=0$

$$\Rightarrow x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$$

$$\overline{\quad\quad}$$
$$2(1)$$

$$\Rightarrow x = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

If x is real, $|A| \neq 0$

$$\therefore \rho(A) = 3$$

6. Determine the values of p such that the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ p & 2 & 2 & 2 \\ 9 & 9 & p & 3 \end{bmatrix}$ is 3

Soln :- If the rank of A is 3 then $|A|$ must be zero and atleast one minor of order 3 must be non-zero

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ p & 2 & 2 & 2 \\ 9 & 9 & p & 3 \end{bmatrix} \xrightarrow[C_3 + C_1]{C_2 - C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ p & 2-p & p+2 & 2 \\ 9 & 0 & p+9 & 3 \end{bmatrix}$$

Now the determinant of this matrix is equal to zero

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ p & 2-p & p+2 & 2 \\ 9 & 0 & p+9 & 3 \end{vmatrix} = 1\,(\text{cofacter of } 1)$$

$$= 1(-1)^{1+1} \begin{vmatrix} 0 & 1 & 1 \\ 2-p & p+2 & 2 \\ 0 & p+9 & 3 \end{vmatrix} = 0$$

$$\Rightarrow (p+6)(p-2) = 0$$

$$\Rightarrow p = -6 \text{ or } p = 2$$

when $p = 2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ 2 & 0 & 4 & 2 \\ 9 & 0 & 11 & 3 \end{bmatrix}$$

consider minor of order 3

$$\begin{vmatrix} 4 & 1 & 1 \\ 2 & 4 & 2 \\ 9 & 11 & 3 \end{vmatrix}$$

$$= 4(12-22) - 1(6-18) + 1(22-36)$$

$$= -40 + 12 - 14 \neq 0$$

$\therefore$ when $p = 2$, $\rho(A) = 3$

Similarly check for $p = -6$

7. If $A = \begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$ is the given square matrix of order 3, find the values of k for which rank of A is less than 3. Also find the ranks for those values of k.

Soln:- $A = \begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$

$\rho(A) < 3$ if $|A| = 0$

$R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix}$$

for the rank to be less than 3, $|A| = 0$

(expanding over 1st column)

$= 1\,(\text{cofactor of } 1)$

$= 1\,(-1)^{1+1} \begin{vmatrix} k-8 & -5k \\ k-2 & -k+2 \end{vmatrix}$

$= 4k^2 - 16$

$\therefore 4k^2 - 16 = 0 \Rightarrow k^2 = 4 \Rightarrow k = \pm 2$

$\therefore$ Rank(A) < 3 when $k = \pm 2$

Case (i) $k = 2$

$$A = \begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

consider a minor of order 2

$$\begin{vmatrix} 1 & 6 \\ 0 & -6 \end{vmatrix} = -6 \neq 0$$

$\therefore$ Rank (A) = 2 when $k = 2$

(ii) when $k = -2$ (H.W.)

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ECHELON FORM OR CANONICAL FORM OF A MATRIX:

Definition: If a matrix A is reduced to a matrix B by using elementary row transformations alone, then B is said to be row equivalent to A.

A $\xrightarrow{\text{Row transf}}$ Row Echelon form

The **Echelon form** or **Canonical form** of a matrix A is a row equivalent matrix of rank 'r' in which

(a) One or more elements of each of the first r rows are non – zero while all other rows have only zero elements, (i.e all zero rows, if any, are placed at the bottom of the matrix so that the first r rows form an upper triangular matrix).

(b) The number of zero before the first non – zero element in a row is less than the number of such zeros in the next row.

In short, by performing only row transformations, a given matrix that is reduced to an **upper triangular form** is called its **Echelon form**.

elements below the diagonal are zero

Note: Rank of a given matrix is equal to the number of non – zero rows in the Echelon form.

For example, the matrix $\begin{bmatrix} 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ of order 4×5 is the Echelon form.

$\rho(A) = 2$

(a) First 2 rows contain at least one non – zero elements while other (i.e 3[rd] and 4[th]) rows have only zero elements.

(b) The number of zeros before the first non – zero element in the first row is one while the number of zeros before the first non – zero element in the second row is two.

Further, there are two non – zero rows in this Echelon form. Hence rank of the matrix is 2.

SOME SOLVED EXAMPLES:

1. Reduce the matrix $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ to Echelon Forms and hence find the ranks.

Soln:-

$$A = \begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix} \xrightarrow{R_{14}} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 3 & 4 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_4 - 3R_1]{R_2 - 2R_1,\ R_3 + R_1} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 6 & -1 & 12 \\ 0 & -3 & 8 & 1 \\ 0 & 7 & -5 & 10 \end{bmatrix} \xrightarrow{R_2 - R_4} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & -3 & 8 & 1 \\ 0 & 7 & -5 & 10 \end{bmatrix}$$

$$\xrightarrow[R_4 + 7R_2]{R_3 - 3R_2} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 23 & 24 \end{bmatrix}$$

$$\xrightarrow{6R_3} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & -24 & -30 \\ 0 & 0 & 23 & 24 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_4} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & 23 & 24 \end{bmatrix}$$

$$\xrightarrow{R_4 + 23R_3} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & -1 & -6 \end{bmatrix}$$

$$\xrightarrow{R_4+23R_3} \begin{bmatrix} 0 & -1 & 4 & 2 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & -114 \end{bmatrix}$$

This is the echelon form of matrix

There are 4 non zero rows

$\therefore \rho(A) = 4.$

(2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

2. Find $\rho(A)$ where $A = \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix}$ for $x \neq y \neq z$