TYPES OF MATRICES

Tuesday, October 5, 2021 8:22 PM

MATRICES

DEFINITION: A system of mn elements (not necessarily distinct) arranged in a rectangular formation of m rows and n columns enclosed by a pair of square brackets is called as m by n matrix or a matrix of order m by n ; which is written as m $\times$n matrix and usually denoted by capital letters.

The matrix can also be expressed in the form: $A = \left[a_{ij}\right]_{m\times n}$ where a_{ij} is the element of i^{th} row and j^{th} column, written as $(i,j)^{th}$ element of the matrix A, $i = 1,2,3, \dots m$; $j = 1,2,3, \dots n$.

TYPES OF MATRICES:

1. **Square Matrix:** In a matrix when the number of rows is same as the number of columns (i.e m = n) then the matrix is called a **square matrix of order n**. (Sometimes it is also called an n – rowed matrix).

In a square matrix the element a_{ij}, where $i = j$, is called a **diagonal element**, i.e., $a_{11}, a_{22}, a_{33}, \dots a_{nn}$ are diagonal elements.

Trace of a Matrix: The sum of the diagonal elements of a square matrix A is called the **trace of A**.

If $A = \left[a_{ij}\right]_{n\times n}$, then trace of $A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots\dots\dots + a_{nn}$

A determinant of a square matrix is such that its elements are same as the corresponding place of a square matrix A, and is denoted by $|A|$ (read as determinant of A). The determinant of A has a numerical value whereas the matrix A is just the arrangement of elements in rows and columns.

Singular Matrix and Non - Singular Matrix: $A = \left[a_{ij}\right]_{n\times n}$ is a given square matrix. If the determinant of A is zero then A is called a singular matrix. i.e., A is singular if and only if $|A| = 0$ *and if* $|A| \neq 0$ then A is said to be non – singular

2. **Row Matrix:** If a matrix has only one row (i.e., m = 1) and any number of columns, then it is called a row matrix or a row – vector which can be expressed as $A = \left[a_{11}\ a_{12}\ a_{13} \dots . a_{1n}\right]_{1\times n}$

3. **Column Matrix:** A matrix having only one column (i.e., n = 1) and any number of rows is called a column matrix or a column – vector and can be expressed as $A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix}_{m\times 1}$

4. **Zero or Null Matrix:** A matrix, rectangular or square, whose all elements are zero, is called zero matrix or null matrix and is denoted by O.

5. **Triangular Matrix:** In a square matrix A = $\left[a_{ij}\right]_{n\times n}$, if the element $a_{ij} = 0$ *for* $i > j$, is called **upper triangular matrix** and if the element $a_{ij} = 0$ for $i < j$, is called **lower triangular matrix.**

6. **Diagonal Matrix:** A square matrix, in which all elements except the diagonal elements are zero, is called a diagonal matrix and it is denoted by D.

Thus the matrix $D = \left[a_{ij}\right]_{n\times n}$ is called a diagonal matrix, if $a_{ij} = 0$ *for* $i \neq j$.

If $d_1, d_2, d_3, \dots . . d_n$ are diagonal elements then the diagonal matrix may be expressed as $D = diag\ \{d_1, d_2, d_3, \dots d_n\}$

7. **Scalar Matrix:** If in a diagonal matrix all diagonal elements are equal then it is called a scalar matrix. i.e., $a_{11} = a_{22} = a_{33} = \cdots = a_{nn} = k$, where k is any scalar.

8. **Unit or Identity Matrix:** If in a diagonal matrix all diagonal elements are unity then it is called a unit or identity matrix of order n. i.e., $a_{11} = a_{22} = a_{33} = \cdots = a_{nn} = 1$, and it is denoted by I_n.

8. Unit or Identity Matrix: If in a diagonal matrix all diagonal elements are unity then it is called a unit or identity matrix of order n. i.e., $a_{11} = a_{22} = a_{33} = \cdots = a_{nn} = 1$, and it is denoted by I_n.

9. Transpose of a Matrix: Let $A = [a_{ij}]_{m\times n}$ be a given $m \times n$ matrix. A matrix obtained by interchanging rows & columns of A, is called a transpose of the matrix A and is denoted by A' or A^T. Thus $A^T = [a_{ji}]_{n\times m}$

$A^t = A$

10. Symmetric Matrix: In a square matrix $A = [a_{ij}]_{n\times n}$, if $a_{ij} = a_{ji}$ for all i and j, then it is called symmetric matrix. i.e., The matrix A is symmetric if and only if $A = A^T$

$a_{ij} = a_{ji}$

For example, (i) $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ **(ii)** $B = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 4 & 5 \\ -2 & 5 & 6 \end{bmatrix}$

$a_{ij} = -a_{ji}$

11. Skew – Symmetric Matrix: In a square matrix $A = [a_{ij}]_{n\times n}$ if $a_{ij} = -a_{ji}$ for all i and j, then it is called a skew symmetric matrix. i.e., The matrix A is skew symmetric if and only if $A = -A^T$

$A^t = -A$

$a_{ii} = -a_{ii}$

$2a_{ii} = 0$

$\Rightarrow a_{ii} = 0$

From the definition, it follows that for diagonal elements
$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \; or \; a_{ii} = 0 \; for \; all \; i.$
i.e., diagonal elements of skew – symmetric matrix are all zero.

For example: $A = \begin{bmatrix} 0 & 7 & -2 \\ -7 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$ is a skew symmetric matrix.

12. Conjugate of a Matrix: $A = [a_{ij}]_{n\times n}$ is a given m by n matrix with some of its elements being complex numbers. A matrix obtained from the given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$.

For example: If $A = \begin{bmatrix} 1+2i & 3-5i & -7 \\ -4i & 6 & 9+i \end{bmatrix}$ $then \; \bar{A} = \begin{bmatrix} 1-2i & 3+5i & -7 \\ 4i & 6 & 9-i \end{bmatrix}$

Note: If A is a matrix over the field of real numbers, then obviously $\bar{A}$ coincides with A.

$A \to A^t \to \overline{A^t}$

$A \to \bar{A} \to (\bar{A})^t = A^\theta$

13. Transposed Conjugate of a Matrix: The transpose of the conjugate of a given matrix A is called transposed conjugate of A and is denoted by $A^\theta \; or \; A^*$

For example: If $A = \begin{bmatrix} 1+2i & 3-5i & -7 \\ -4i & 6 & 9+i \end{bmatrix}_{2\times 3}$ $then \; \bar{A} = \begin{bmatrix} 1-2i & 3+5i & -7 \\ 4i & 6 & 9-i \end{bmatrix}_{2\times 3}$

$\therefore A^\theta = \begin{bmatrix} 1-2i & 4i \\ 3+5i & 6 \\ -7 & 9-i \end{bmatrix}_{3\times 2}$

Obviously the conjugate of the transpose of A is same as the transpose of the conjugate of A.

i.e., $A^\theta = (\bar{A})^T = \overline{(A^T)}$

$a_{ij} = \overline{a_{ji}}$

$A^\theta = A$

14. Hermitian Matrix: In a square matrix $A = [a_{ij}]_{n\times n}$, $if \; a_{ij} = \bar{a}_{ji}$ for all i and j, then it is called a Hermitian matrix, i.e., The matrix A is Hermitian if and only if $A = A^\theta$.

$a_{ii} = a_{ii}$

If A is Hermitian matrix, then for all diagonal elements, we have $a_{ii} = \bar{a}_{ii}$ for all i, by definition.
Let $a_{ii} = x + iy, then \; a_{ii} = \bar{a}_{ii} \Longrightarrow x + iy = x - iy \Rightarrow iy = -iy \Rightarrow 2iy = 0$
i.e The imaginary part is zero. $\therefore a_{ii}$ is real for all i.
Thus every diagonal element of a Hermitian matrix must be real.

For example: **(i)** $A = \begin{bmatrix} a & b-ic \\ b+ic & d \end{bmatrix}$ **(ii)** $B = \begin{bmatrix} 1 & 2+3i & 4-5i \\ 2-3i & 0 & 3+4i \\ 4+5i & 3-4i & 2 \end{bmatrix}$

$a_{ij} = -\overline{a_{ji}}$

Note: A Hermitian matrix over the field of real numbers is nothing but a real symmetric matrix

$A^\theta = -A$

15. Skew – Hermitian Matrix: In a square matrix $A = [a_{ij}]$ $if \; a_{ij} = -\bar{a}_{ji}$ for all i and j, then it is called a

Note: A Hermitian matrix over the field of real numbers is nothing but a real symmetric matrix

$A^\theta = -A$

15. **Skew – Hermitian Matrix:** In a square matrix $A = \left[a_{ij}\right]_{n\times n}$, $if\ a_{ij} = -\bar{a}_{ji}$ for all i and j, then it is called a skew – Hermitian matrix, i.e., The matrix A is skew – Hermitian if and only if $A = -A^\theta$.

$a_{ij} = -\overline{a_{ij}}$

If A is skew – Hermitian matrix, then for all diagonal elements,
we have $a_{ii} = -\bar{a}_{ii}$ for all i, by definition. i,e., $a_{ii} + \bar{a}_{ii} = 0$ for all i.
Let $a_{ii} = x + iy$ then $a_{ii} + \bar{a}_{ii} = 0 \Rightarrow (x + iy) + (x - iy) = 0 \Rightarrow 2x = 0$
i.e., The real part is zero. i.e., a_{ii} must be either zero or purely imaginary number.
Thus the diagonal of a skew – Hermitian matrix must be either zero or purely imaginary number.

For example: **(i)** $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 3i \end{bmatrix}$ **(ii)** $B = \begin{bmatrix} i & -2+i & -2i \\ 2+i & 0 & 3+2i \\ -2i & -3+2i & 4i \end{bmatrix}$

Note: A skew –Hermitian matrix over the field of real numbers is nothing but a real skew – symmetric matrix.

Symm
$a_{ij} = a_{ji}$
$A^t = A$

Skew Symm
$a_{ij} = -a_{ji}$
$A^t = -A$
All diagonal elements are zero

Hermitian
$a_{ij} = \overline{a_{ji}}$
$A^\theta = A$
All diagonal elements are real

Skew Hermitian
$a_{ij} = -\overline{a_{ji}}$
$A^\theta = -A$
All diagonal elements are purely imaginary or zero.

OPERATIONS ON MATRICES:

1. **Equality of Two Matrices:** Two matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are said to be equal if

(i) They are of the same size, i.e A and B have same order.

(ii) $a_{ij} = b_{ij}$ for all the values of i and j.

Thus, $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 0 \end{bmatrix}$ Then, $a = 3, b = 4, c = 5, d = 2, e = -1, f = 0$

2. **Summation and Subtraction of Matrices:** Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be two matrices of same order $m \times n$, then their sum(or difference), denoted by $A + B(or\ A - B)$, is defined to be the matrix of the same order $m \times n$ obtained by adding(or subtracting) the corresponding elements of A and B.

Thus $A \pm B = \left[a_{ij} \pm b_{ij}\right]_{m\times n}$

For example, If $A = \begin{bmatrix} 1 & 0 & -3 \\ 5 & 4 & 6 \end{bmatrix}_{2\times 3}$ $and\ B = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 0 \end{bmatrix}_{2\times 3}$

then $A + B = \begin{bmatrix} 4 & 4 & 2 \\ 7 & 3 & 6 \end{bmatrix}_{2\times 3}$ $and\ A - B = \begin{bmatrix} -2 & -4 & -8 \\ 3 & 5 & 6 \end{bmatrix}_{2\times 3}$

3. **A scalar Multiple of a Matrix:** Let $A = \left[a_{ij}\right]$ be a matrix of order $m \times n$ and k be any scalar.

A matrix obtained from A by multiplying each of its elements by k is called the scalar multiple of A by k and is denoted by kA. Thus $kA = \left[ka_{ij}\right]_{m\times n}$

For example, If $A = \begin{bmatrix} 7 & 3 & 6 \\ 1 & 0 & -2 \end{bmatrix}_{2\times 3}$ $and\ k = 3, kA = \begin{bmatrix} 21 & 9 & 18 \\ 3 & 0 & -6 \end{bmatrix}_{2\times 3}$

4. **Multiplication of Two Matrices:** The product AB of two matrices $A = \left[a_{ij}\right]$ $and\ B = \left[b_{ij}\right]$ exists if and only if **the number of columns in A is equal to the number of rows in B.**

Thus two matrices $A = \left[a_{ij}\right]$ $and\ B = \left[b_{ij}\right]$ are said to be conformable for multiplication if A is of order $m \times p$ and B is of order $p \times n$. Then the product $AB = \left[c_{ij}\right]$ is a matrix of order $m \times n$, where

$$c_{ij} = a_{i1}.b_{1j} + a_{i2}.b_{2j} + a_{i3}.b_{3j} + \cdots a_{ip}.b_{pj} = \sum_{k=1}^{p} a_{ik}.b_{kj}, \text{ the } (i,j)^{th} \text{ element of AB}$$

For instance, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$ $and\ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$

then $C = AB = \begin{bmatrix} a_{11}.b_{11} + a_{12}.b_{21} + a_{13}.b_{31} & a_{11}.b_{12} + a_{12}.b_{22} + a_{13}.b_{32} \\ a_{21}.b_{11} + a_{22}.b_{21} + a_{23}.b_{31} & a_{21}.b_{12} + a_{22}.b_{22} + a_{23}.b_{32} \end{bmatrix}$

$$\begin{bmatrix} \cdots \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \qquad \begin{bmatrix} \cdots \\ b_{31} & b_{32} \end{bmatrix}_{3\times2}$$

then $C = AB = \begin{bmatrix} a_{11}.b_{11} + a_{12}.b_{21} + a_{13}.b_{31} & a_{11}.b_{12} + a_{12}.b_{22} + a_{13}.b_{32} \\ a_{21}.b_{11} + a_{22}.b_{21} + a_{23}.b_{31} & a_{21}.b_{12} + a_{22}.b_{22} + a_{23}.b_{32} \\ a_{31}.b_{11} + a_{32}.b_{21} + a_{33}.b_{31} & a_{31}.b_{12} + a_{32}.b_{22} + a_{33}.b_{32} \end{bmatrix}_{3\times2}$

For example: If $A = \begin{bmatrix} 7 & -3 & 6 \\ 5 & 4 & -1 \\ 2 & 1 & 3 \end{bmatrix}_{3\times3}$ $and\ B = \begin{bmatrix} 3 & 4 \\ 2 & -1 \\ 5 & 7 \end{bmatrix}_{3\times2}$

then $AB = \begin{bmatrix} 21-6+30 & 28+3+42 \\ 15+8-5 & 20-4-7 \\ 6+2+15 & 8-1+21 \end{bmatrix}_{3\times2} = \begin{bmatrix} 45 & 73 \\ 18 & 9 \\ 23 & 28 \end{bmatrix}_{3\times2}$

Note: **(i)** Whenever the matrix AB exists, it is not necessary that BA should also exists

(ii) Whenever the matrices AB and BA both exist, it is not necessary that $AB = BA$.

(iii) $AB = O$ does not necessarily mean $A = O, B = O$

(iv) $AB = AC$ does not necessarily mean $B = C$

$AB = 0$

$A = 0$ or $B = 0$

$\times$

$A \neq 0$, $B \neq 0$

but $AB = 0$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$

$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Theorems on Trace of A Matrix:

Let A and B be two square matrices of order n and k be a scalar, then

(i) Trace of (kA) = k(Trace of A)

(ii) Trace of $(A + B)$ = (Trace of A) + (Trace of B)

(iii) Trace of (AB) = Trace of (BA)

Theorems on Transposes and Transposed Conjugates Of Matrices:

Theorem: If A^T and B^T be the transposes of A and B respectively, then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T, A\ and\ B$ being of the same size;

(iii) $(kA)^T = kA^T, k$ is any scalar.

(iv) $(AB)^T = B^T A^T$, A and B being conformable for multiplication.

(v) $(ABC)^T = C^T B^T A^T$

Theorem: If $\overline{A}\ and\ \overline{B}$ be the conjugates of A and B respectively, then

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{(A + B)} = \overline{A} + \overline{B}$, A and B being of the same size

(iii) $\overline{(kA)} = \overline{k}\,\overline{A}$, k being any scalar

(iv) $\overline{(AB)} = \overline{A}\,\overline{B}$, A and B being conformable for multiplication.

Theorem: If A^θ and B^θ be the transposed conjugate matrices of A and B respectively, then

(i) $(A^\theta)^\theta = A$

(ii) $(A + B)^\theta = A^\theta + B^\theta$, A and B being of the same size

(iii) $(kA)^\theta = \overline{k}A^\theta$, k is any scalar

(iv) $(AB)^\theta = B^\theta A^\theta$, A and B being conformable for multiplication.

THEOREMS ON SYMMETRIC AND HERMITIAN MATRICES:

Proof :- Let A be any square matrix

Theorem (1): Show that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew – symmetric matrix.

Let $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$

$= P + Q$

TPT P is symmetric ie $P^t = P$

$P^t = \frac{1}{2}(A^t + (A^t)^t)$

Tpt P is symmetric ...

$$P^t = \left[\frac{1}{2}(A+A^t)\right]^t = \frac{1}{2}(A^t+(A^t)^t)$$

$$= \frac{1}{2}(A^t+A) = \frac{1}{2}(A+A^t) = P$$

Tpt Q is skew symmetric ie $Q^t = -Q$

$$Q^t = \left[\frac{1}{2}(A-A^t)\right]^t = \frac{1}{2}(A^t-(A^t)^t)$$

$$= \frac{1}{2}(A^t-A) = -\frac{1}{2}(A-A^t) = -Q$$

$\therefore A = P+Q$ where P is symmetric and Q is skew symmetric

To prove uniquenss

Let $A = R+S$ where R is symmetric and S is skew symmetric

$$\therefore R^t = R \quad \& \quad S^t = -S$$

Now
$$A+A^t = (R+S)+(R+S)^t$$
$$= R+S+R^t+S^t$$
$$= R+S+R-S$$
$$= 2R$$

$$\therefore R = \frac{1}{2}(A+A^t) = P$$

$$A-A^t = (R+S)-(R+S)^t = (R+S)-(R^t+S^t)$$
$$= (R+S)-(R-S) = 2S$$

$$\therefore S = \frac{1}{2}(A-A^t) = Q$$

Hence, the representation $A = P+Q$ is unique.

Hence proved.

Theorem (2): Show that every square matrix can be uniquely expressed as the sum of a Hermitian matrix and a

Hence proved

Theorem (2): Show that every square matrix can be uniquely expressed as the sum of a Hermitian matrix and a Skew – Hermitian matrix. (H.W.)

Proof: Let A be any square matrix

let $A = \frac{1}{2}(A+A^\theta) + \frac{1}{2}(A-A^\theta)$

$= P + Q$

Tpt P is Hermitian & Q is skew-Hermitian

11/15/2021 1:15 PM

Theorem (3): Show that every square matrix A can be uniquely expressed as $P + iQ$ where P and Q are Hermitian matrices

Proof:- Let A be any square matrix

Let $A = \frac{1}{2}(A+A^\theta) + i\left[\frac{1}{2i}(A-A^\theta)\right]$

$= P + iQ$

where $P = \frac{1}{2}(A+A^\theta)$, $Q = \frac{1}{2i}(A-A^\theta)$

Tpt P is Hermitian ie $P^\theta = P$

$P^\theta = \left[\frac{1}{2}(A+A^\theta)\right]^\theta = \frac{1}{2}(A+A^\theta)^\theta = \frac{1}{2}(A^\theta + (A^\theta)^\theta)$

$= \frac{1}{2}(A+A^\theta) = P$

Tpt Q is Hermitian, $Q^\theta = Q$

$Q^\theta = \left[\frac{1}{2i}(A-A^\theta)\right]^\theta = \frac{1}{-2i}(A-A^\theta)^\theta$ (taking conjugate of $\frac{1}{2i}$)

$\frac{-}{2i}(A^\theta - A)$

$= \frac{1}{2i}(A-A^\theta) = Q$

$\therefore A = P + iQ$ where P & Q both are Hermitian.

To prove uniqueness, we assume that

$A = R + iS$ where R & S are Hermitian

$$A + A^\theta = (R+iS) + (R+iS)^\theta = (R+iS) + (R^\theta - iS^\theta)$$

$$= (R+iS) + (R-iS) = 2R$$

$$\therefore R = \frac{1}{2}(A+A^\theta) = P$$

Also, $A - A^\theta = (R+iS) - (R+iS)^\theta = (R+iS) - (R^\theta - iS^\theta)$

$$= (R+iS) - (R-iS) = 2iS$$

$$\therefore S = \frac{1}{2i}(A - A^\theta) = Q$$

Hence the representation is unique.

Theorem (4): Prove that every Hermitian matrix A can be written as $A = B + iC$ where B is real symmetric and C is real skew – symmetric.

Proof :- let A be any Hermitian matrix

Then $A^\theta = A$ ——— (1)

$$\text{let } A = \frac{1}{2}(A+\bar{A}) + i\left[\frac{1}{2i}(A-\bar{A})\right] = B + iC$$

We know that if $z = x+iy$ is a complex number then $\frac{1}{2}(z+\bar{z})$ and $\frac{1}{2i}(z-\bar{z})$ both are real

$\therefore$ B & C both are real matrix

Now we show that B is symmetric and C is skew symmetric.

$$B^t = \left[\frac{1}{2}(A+\bar{A})\right]^t = \frac{1}{2}(A^t + (\bar{A})^t) = \frac{1}{2}(A^t + A^\theta)$$

from (1) $\rightarrow$ $A^\theta = A \Rightarrow \overline{(A^t)} = A \Rightarrow A^t = \bar{A}$

$\therefore B^t = \frac{1}{2}(\bar{A} + A) = B \quad \therefore B$ is symmetric

$$C^t = \left[\frac{1}{2i}(A - \bar{A})\right]^t = \frac{1}{2i}\left(A^t - (\bar{A})^t\right) = \frac{1}{2i}\left(A^t - A^\theta\right)$$

$$= \frac{1}{2i}(\bar{A} - A) = \frac{-1}{2i}(A - \bar{A}) = -C$$

$\therefore$ C is skew symmetric

$\therefore A = B + iC$ where B is real symmetric and C is real skew symmetric

uniqueness part (H.W.)

Theorem (5): Prove that every Skew – Hermitian matrix A can be written as $B + iC$ where, B is real skew – symmetric and C is real symmetric matrix.

Let A be any skew-Hermitian matrix

$$\therefore A^\theta = -A \quad \text{——} \quad (1)$$

Let $A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$

$= B + iC$

H.W.

SOME SOLVED EXAMPLES:

1. If A is symmetric matrix, then prove that A^n is also symmetric. Is this result valid if A is skew – symmetric ?

proof :- Let A be symmetric matrix

$$\therefore A^t = A$$

TPT A^n is symmetric ie $(A^n)^t = A^n$

$$(A^n)^t = (\underbrace{A A A \cdots A}_{n \text{ times}})^t \qquad \left[(AB)^t = B^t A^t\right]$$

$$= \underbrace{A^t A^t A^t \cdots A^t}_{n \text{ times.}} = A A A \cdots A \quad (A^t = A)$$

$(A^n)^t = A^n \quad \therefore A^n$ is symmetric.

Now if A is skew symmetric ie $A^t = -A$

$$(A^n)^t = (\underbrace{A \cdots\cdots A A}_{n \text{ times}})^t = \underbrace{A^t A^t A^t \cdots A}_{n \text{ times}}$$

$$= \underbrace{(-A)(-A)\cdots\cdots(-A)}_{n \text{ times}} = (-1)^n A^n$$

$\therefore (A^n)^t = (-1)^n A^n$

If n is even $(A^n)^t = A^n \Rightarrow A^n$ is symmetric

If n is odd $(A^n)^t = -A^n \Rightarrow A^n$ is skew-symm.

2. If A and B are Hermitian matrices then prove that $(AB + BA)$ is Hermitian and $(AB - BA)$ is skew – Hermitian.

<u>Proof</u> :- A and B are Hermitian

$\therefore A^\theta = A$ and $B^\theta = B$

TPT $(AB+BA)$ is Hermitian ie $(AB+BA)^\theta = AB+BA$

$$(AB+BA)^\theta = (AB)^\theta + (BA)^\theta = B^\theta A^\theta + A^\theta B^\theta$$

$$= BA + AB = AB + BA$$

$\therefore AB+BA$ is Hermitian

Similarly we can prove that $AB - BA$ is skew Hermitian (H.w.)

3. If A is any square matrix, then show that $A + A^T$ is symmetric and $A - A^T$ is skew – symmetric <u>H.w.</u>

$(A+A^t)^t = A^t + (A^t)^t = A + A^t \rightarrow$ Symm.

$(A-A^t)^t = A^t - (A^t)^t = A^t - A = -(A-A^t)$

$\rightarrow$ Skew-Symm.

4. If A is a Hermitian matrix, show that iA is skew – Hermitian and if A is a skew- Hermitian matrix then show that iA is Hermitian.

Proof :- Let A be Hermitian $\Rightarrow A^\theta = A$

$(iA)^\theta = -iA^\theta = -iA \Rightarrow iA$ is skew Hermitian

Similarly, we can prove the other part

5. Show that the matrix $B^T AB$ is symmetric or skew – symmetric accordingly when A is symmetric or skew – symmetric matrix.

Soln :- (i) Let A be symmetric matrix

Then $A^t = A$

Now $(B^tAB)^t = B^t A^t (B^t)^t = B^t A B$

$\Rightarrow B^tAB$ is symmetric

(ii) If A is skew symmetric

Then $A^t = -A$

Now $(B^tAB)^t = B^t A^t (B^t)^t = B^t(-A)B$

$= -(B^tAB)$

$\therefore B^tAB$ is skew symmetric

6. If A and B are symmetric matrices, then show that AB is symmetric if and only If A and B commute

$\Rightarrow AB = BA$

Proof :- It is given that A & B are symmetric

$\therefore A = A^t$ and $B = B^t$

(I) If A & B commute ie $AB = BA$

Then TPT AB is symmetric

$(AB)^t = B^tA^t = BA = AB \Rightarrow AB$ is Symm.

(II) conversely, Let AB be symmetric then

To prove that A and B commute

we have $AB = (AB)^t$

$\therefore AB = B^t A^t$

$\therefore AB = BA \Rightarrow A \,\&\, B$ commute.

7. Express the matrix $\begin{bmatrix} 1 & 6 & 6 \\ -2 & -2 & -3 \\ 2 & 7 & 5 \end{bmatrix}$ as the sum of symmetric and skew – symmetric matrices

Sol^n:- $A = \begin{bmatrix} 1 & 6 & 6 \\ -2 & -2 & -3 \\ 2 & 7 & 5 \end{bmatrix}$

Let $A = P + Q$ where $P = \frac{1}{2}(A + A^t)$

$Q = \frac{1}{2}(A - A^t)$

$$\therefore A^t = \begin{bmatrix} 1 & -2 & 2 \\ 6 & -2 & 7 \\ 6 & -3 & 5 \end{bmatrix}$$

$$\therefore P = \frac{1}{2}(A + A^t) = \frac{1}{2}\left\{ \begin{bmatrix} 1 & 6 & 6 \\ -2 & -2 & -3 \\ 2 & 7 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 6 & -2 & 7 \\ 6 & -3 & 5 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 2 & -2 & 2 \\ 4 & 2 & 5 \end{bmatrix} \text{ which is symmetric}$$

$$Q = \frac{1}{2}(A - A^t) = \frac{1}{2}\left\{ \begin{bmatrix} 1 & 6 & 6 \\ -2 & -2 & -3 \\ 2 & 7 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 6 & -2 & 7 \\ 6 & -3 & 5 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 & 4 & 2 \\ -4 & 0 & -5 \\ -2 & 5 & 0 \end{bmatrix} \text{ which is skew symmetric}$$

8. Express the matrix $\begin{bmatrix} 3+4i & 1-i & 2+3i \\ 1+i & 4-5i & 1 \\ 5 & 3 & 3-i \end{bmatrix}$ as the sum of Hermitian and skew – Hermitian matrices

Theorem-2, $A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$

$$A = \begin{bmatrix} 3+4i & 1-i & 2+3i \\ 1+i & 4-5i & 1 \\ 5 & 3 & 3-i \end{bmatrix}$$

$$\overline{A^t} = \begin{bmatrix} 3+4i & 1+i & 5 \\ 1-i & 4-5i & 3 \\ 2+3i & 1 & 3-i \end{bmatrix}$$

$$A^\theta = \overline{A^t} = \begin{bmatrix} 3-4i & 1-i & 5 \\ 1+i & 4+5i & 3 \\ 2-3i & 1 & 3+i \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^\theta)$$

$$Q = \frac{1}{2}(A - A^\theta)$$

9. Express the Hermitian matrix $\begin{bmatrix} 2 & 4+i & 8+3i \\ 4-i & 12 & 6+i \\ 8-3i & 6-i & 2 \end{bmatrix}$ as $B + iC$ where B is real symmetric and C is real skew symmetric.

Based on Theorem-4 $A = B + iC$

$$= \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$$

10. Express the skew – Hermitian matrix $\begin{bmatrix} 3i & -2+i & 4+2i \\ 2+i & -4i & 3+5i \\ -4+2i & -3+5i & 7i \end{bmatrix}$ as $P + iQ$ where P is real skew – symmetric and Q is real symmetric.

Theorem-5, $A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$

$$= P + iQ$$

11/17/2021 2:29 PM

11. If $f(x) = x^2 - 5x + 6$, find $f(A)$, where $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ Is $f(A)$ non – singular?

Soln: $f(x) = x^2 - 5x + 6$

$\therefore f(A) = A^2 - 5A + 6I$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\therefore f(A) = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Now $|f(A)| = \begin{vmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{vmatrix} = 1(-4+40) + 1(-4-50) - 3(-4-5)$

$\therefore |f(A)| = 9 \neq 0$

$\therefore$ $f(A)$ is non singular

12. A is a skew – symmetric matrix of odd order then, A is singular i.e., $|A| = 0$

Soln: A is skew - symmetric of order n (n is odd)

$\therefore A^t = -A$

$|A^t| = |-A|$ (taking determinant)

$\therefore |A| = |-A|$ $(\because |A^t| = |A|)$

Taking (-1) common from each row in $|-A|$

$|A| = (-1)^n |A|$

If n is odd, $(-1)^n = -1$

$\therefore |A| = -|A| \Rightarrow 2|A| = 0 \Rightarrow |A| = 0$

$\therefore$ A is singular matrix

Note :- If A is a skew-symmetric matrix of odd order then A^{-1} does not exist.

ORTHOGONAL AND UNITARY MATRICES:

Orthogonal Matrix:

Definition: A square matrix A is said to be orthogonal if and only if $AA^T = A^T A = I$

Since $|A^T| = |A|$ and $|A^T A| = |A^T||A|$

$\therefore A^T A = I$, we have $|A|^2 = 1 \Rightarrow |A| = \pm 1$

This implies the following

Note: **(i)** Determinant of an orthogonal matrix is ± 1.

(ii) If A is orthogonal then it is non – singular and the inverse of A is A^T (i.e., $A^{-1} = A^T$)

$AA^t = A^t A = I$

$|AA^t| = |I|$

$|A||A^t| = |I|$

$|A||A| = 1$

$|A|^2 = 1$

$|A| = \pm 1$

Theorem: If A and B are n – rowed orthogonal matrices then AB and BA are both orthogonal.

Proof :- Let A be B be n-rowed orthogonal matrices

$\therefore AA^t = A^t A = I$ and $BB^t = B^t B = I$

As A and B are square matrices of order n AB and BA are also square matrices of order n.

To show that AB is orthogonal,

$(AB)(AB)^t = (AB)(B^t A^t)$

$= A(BB^t)A^t = A(I)A^t = AA^t = I$

$\therefore$ AB is orthogonal matrix

$\therefore$ Similarly we can prove that BA is also orthogonal matrix.

1. Show that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal and find its inverse.

Soln :- Let $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$$AA^t = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore AA^t = I$

$\therefore A$ is orthogonal matrix

$$\therefore A^{-1} = A^t = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

H.W.

Show that the matrix $\begin{bmatrix} cos\emptyset cos\theta & sin\emptyset & cos\emptyset sin\theta \\ -sin\emptyset cos\theta & cos\emptyset & -sin\emptyset sin\theta \\ -sin\theta & 0 & cos\theta \end{bmatrix}$ is orthogonal and find its inverse.

2. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ is orthogonal, find a,b,c. Also find the inverse of A.

Soln:- Since A is orthogonal,

$AA^t = I$

$$\frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{9}\begin{bmatrix} 5+a^2 & 4+ab & -2+ac \\ 4+ab & 5+b^2 & 2+bc \\ -2+ac & 2+bc & 8+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

comparing both sides,

comparing both sides,

$$\frac{5+a^2}{9}=1 \quad , \quad \frac{5+b^2}{9}=1 \quad , \quad \frac{8+c^2}{9}=1$$

$$\frac{4+ab}{9}=0 \quad , \quad \frac{-2+ac}{9}=0 \quad , \quad \frac{2+bc}{9}=0$$

$$\frac{5+a^2}{9}=1 \Rightarrow a^2=4 \Rightarrow a=\pm 2$$

$$\frac{5+b^2}{9}=1 \Rightarrow b^2=4 \Rightarrow b=\pm 2$$

$$\frac{8+c^2}{9}=1 \Rightarrow c^2=1 \Rightarrow c=\pm 1$$

$\frac{4+ab}{9}=0 \Rightarrow ab=-4$ → If $a=2$ then $b=-2$; If $a=-2$ then $b=2$

$\frac{-2+ac}{9}=0 \Rightarrow ac=2$ → If $a=2$ then $c=1$; If $a=-2$ then $c=-1$

$(a,b,c)=(2,-2,1)$ or $(-2,2,-1)$

$$\therefore A^{-1}=A^t=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & b & c\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix} \text{ or } \frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}$$

3. Is the following matrix orthogonal? If not, can it be converted into an orthogonal matrix? If yes how?

$$A=\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$$

Soln:- Check whether A is orthogonal

$$AA^t=\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}\begin{bmatrix}-8 & 4 & 1\\ 1 & 4 & -8\\ 4 & 7 & 4\end{bmatrix}$$

$$=\begin{bmatrix}81 & 0 & 0\\ 0 & 81 & 0\\ \end{bmatrix}=81\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \end{bmatrix}$$

$$= \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = 81 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A A^t = 81 I$

$\therefore$ A is not orthogonal.

But we can convert this into an orthogonal matrix

$$\frac{1}{81}(AA^t) = I$$

$$\left(\frac{1}{9}A\right)\left(\frac{1}{9}A^t\right) = I$$

$$B\,B^t = I$$

$\therefore B = \frac{1}{9}A$ is the required orthogonal matrix.

11/22/2021 1:10 PM

Unitary Matrix:

Definition: A square matrix A is said to be unitary if and only if $AA^\theta = A^\theta A = I$

Since $|A^\theta| = |\bar{A}|$ and $|A^\theta A| = |A^\theta||A|$

$\therefore$ if $A^\theta A = I, we\ have\ |A||\bar{A}| = 1$

This implies the following

Note: **(i)** The determinant of a unitary matrix is of ~~unit~~ modulus.

(ii) If A is unitary matrix then it is non – singular and the inverse of A is A^θ (i.e $A^{-1} = A^\theta$)

$AA^\theta = A^\theta A = I$

$|AA^\theta| = |I|$

$|A||A^\theta| = 1$

$|A||\bar{A}| = 1$

$z\bar{z} = 1$

$\Rightarrow |z| = 1.$

Theorem: If A and B are n – rowed unitary matrices than AB and BA are both unitary.

Proof: Since A and B are square matrices of order n, AB and BA are also square matrices of order n

TPT AB is unitary

$$(AB)(AB)^\theta = (AB)(B^\theta A^\theta) = A(BB^\theta)A^\theta = A(I)A^\theta$$
$$= AA^\theta = I$$

$\therefore$ AB is unitary matrix

Similarly we can prove for BA as well.

SOME SOLVED EXAMPLES:

1. Prove that the following matrices are unitary and hence find A^{-1}.

(i) $\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ **(ii)** $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Soln:- (i) $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$A^t = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix} \rightarrow A^\theta = \overline{A^t} = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

Now $AA^\theta = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 2+2 & 2-2 \\ 2-2 & 2+2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$\therefore AA^\theta = I$

$\therefore A$ is unitary

$\therefore A^{-1} = A^\theta = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

2. If S is real skew – symmetric matrix and $(I - S)$ is non – singular, then show that $(I + S)(I - S)^{-1}$ is orthogonal.

Soln:- S is skew-symmetric $\Rightarrow S^t = -S$

To show that $(I+S)(I-S)^{-1}$ is orthogonal

let $B = (I+S)(I-S)^{-1}$

ie $BB^t = I$

Now $B^t = \left[(I+S)(I-S)^{-1}\right]^t = \left[(I-S)^{-1}\right]^t (I+S)^t$

$= \left((I-S)^t\right)^{-1}(I^t + S^t)$

$$= \left[(I-S)^t\right]^{-1}(I^t+S^t)$$

$$= \left[(I^t-S^t)\right]^{-1}(I^t+S^t)$$

$$B^t = (I+S)^{-1}(I-S)$$

$$\left(\because \begin{array}{l} I^t = I \\ S^t = -S \end{array}\right)$$

Now $\quad BB^t = (I+S)(I-S)^{-1}(I+S)^{-1}(I-S)$

$$= \underline{(I+S)(I+S)^{-1}}\ \underline{(I-S)^{-1}(I-S)}$$

As. $(I+S)(I-S) = (I-S)(I+S)$

$$\downarrow \qquad\qquad \downarrow$$

$$I^2 - S^2 \qquad I^2 - S^2$$

$$\therefore BB^t = I \cdot I = I$$

$\therefore$ B is orthogonal.