DE MOIVRE'S THEOREM

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DE MOIVRE'S THEOREM:

Statement : For any rational number n the value or one of the values of $(\cos\theta + i\sin\theta)^n = \cos\, n\,\theta + i\, \sin n\,\theta$

$z = e^{i\theta} \quad (z)^{-1} = (e^{i\theta})^{-1} = e^{-i\theta}$

1. If $z = cos\,\theta + i\sin\theta$ then

$\frac{1}{z} = z^{-1} = (\cos\theta + \, i\, sin\,\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - \, i\sin\theta$

i.e. $\frac{1}{z} = \cos\theta - \, i\sin\theta$

2. $(\cos\theta - i\sin\theta)^n = \cos\, n\,\theta - i\, \sin n\,\theta$

For, $(\cos\theta - i\sin\theta)^n = \{\cos\,(-\theta) + i\, \sin\,(-\theta)\}^n$

$= \cos(-n\theta) + isin(-n\theta).$

$= \cos n\,\theta - i\sin\, n\, \theta$

$(\cos\theta \pm i\sin\theta)^n$

$= \cos n\theta \pm i\sin n\theta$

Note : Note carefully that ,

(1) $(\sin\theta + i\cos\theta)^n \neq \sin n\,\theta + i\cos n\,\theta$

But $(\sin\theta + i\cos\theta)^n = [cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)]^n$

$= \cos\, n\left(\frac{\pi}{2} - \theta\right) + i\sin n\,\left(\frac{\pi}{2} - \theta\right)$

n

(2) $(\cos\;\theta + \mathrm{i}\sin\square\;)^n \neq \cos\;n\,\theta + \;i\sin n\;.$

SOME SOLVED EXAMPLES:

1. Simplify $\frac{(\cos 2\theta - i\sin 2\theta)^7(\cos 3\theta + i\sin 3\theta)^5}{(\cos 3\theta + i\sin 3\theta)^{12}(\cos 5\theta - i\sin 5\theta)^7}$

$\cos 2\theta - i\sin 2\theta = (\cos\theta + i\sin\theta)^{-2} = e^{-i2\theta}$

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^{3} = e^{i3\theta}$

$\cos 5\theta - i\sin 5\theta = (\cos\theta + i\sin\theta)^{-5} = e^{-i5\theta}$

Given expression $= \dfrac{(e^{-i2\theta})^7\,(e^{i3\theta})^5}{(e^{i3\theta})^{12}\,(e^{-i5\theta})^7}$

$= \dfrac{e^{-i14\theta} \cdot e^{i15\theta}}{e^{i36\theta}\; e^{-i35\theta}} = \dfrac{e^{i\theta}}{e^{i\theta}}$

$= 1.$

2.

Prove that $\frac{(1+i)^8(\sqrt{3}-i)^4}{(1-i)^4(\sqrt{3}+i)^8} = -\frac{1}{4}$

$\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\, e^{i\pi/4}$

Soln :-

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = \sqrt{2}\,e^{i\pi/4}$$

$$1-i = \sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right) = \sqrt{2}\,e^{-i\pi/4}$$

$$\sqrt{3}-i = 2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right) = 2e^{-i\pi/6}$$

$$\sqrt{3}+i = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) = 2e^{i\pi/6}$$

$$LHS = \frac{(\sqrt{2}e^{i\pi/4})^8(2e^{-i\pi/6})^4}{(\sqrt{2}e^{-i\pi/4})^4(2e^{i\pi/6})^8}$$

$$= \frac{2^4 e^{i2\pi}\cdot 2^4 e^{-i2\pi/3}}{2^2 e^{-i\pi}\cdot 2^8 e^{i4\pi/3}} = \frac{2^8}{2^{10}} e^{i\left(2\pi-\frac{2\pi}{3}+\pi-\frac{4\pi}{3}\right)}$$

$$= \frac{1}{2^2}e^{i(\pi)} = \frac{1}{4}\left[\cos\pi + i\sin\pi\right]$$

$$= \frac{-1}{4} = RHS$$

$\cos\pi = -1$
$\sin\pi = 0$.

3.

Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}}$

$$1+i\sqrt{3} = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) = 2e^{i\pi/3}$$

$$\sqrt{3}-i = 2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right) = 2e^{-i\pi/6}$$

$$\frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}} = \frac{(2e^{i\pi/3})^{16}}{(2e^{-i\pi/6})^{17}} = \frac{2^{16}}{2^{17}}\frac{e^{i16\pi/3}}{e^{-i17\pi/6}}$$

$$= \frac{1}{2}e^{i\left(\frac{16\pi}{3}+\frac{17\pi}{6}\right)}$$

$$= \frac{1}{2}e^{i\left(\frac{49\pi}{6}\right)} = \frac{1}{2}\left[\cos\left(\frac{49\pi}{6}\right)+i\sin\frac{49\pi}{6}\right]$$

$$= \frac{1}{2}\left[\cos\left(8\pi+\frac{\pi}{6}\right)+i\sin\left(8\pi+\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]$$

modulus $= \frac{1}{2}$, principal value of argument $= \frac{\pi}{6}$.

4. Simplify $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n$

Soln :- $1 = \sin^2\alpha+\cos^2\alpha = \sin^2\alpha - i^2\cos^2\alpha$

$$= (\sin\alpha+i\cos\alpha)(\sin\alpha-i\cos\alpha)$$

$$1+\sin\alpha+i\cos\alpha = (\sin\alpha+i\cos\alpha)(\sin\alpha-i\cos\alpha)+(\sin\alpha+i\cos\alpha)$$

$$= (\sin\alpha+i\cos\alpha)[\sin\alpha-i\cos\alpha+1]$$

$$= (\sin\alpha+i\cos\alpha)(1+\sin\alpha-i\cos\alpha)$$

$$\therefore \frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha} = \sin\alpha+i\cos\alpha$$

$$\therefore \left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = (\sin\alpha+i\cos\alpha)^n$$

$$= \left[\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)\right]^n$$

$$= \cos n\left(\frac{\pi}{2}-\alpha\right)+i\sin n\left(\frac{\pi}{2}-\alpha\right)$$

5. If $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ *and* $\bar{z}$ is the conjugate of z prove that $(z)^{10} + (\bar{z})^{10} = 0$.

$$z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$\bar{z} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$$

$$(z)^{10} + (\bar{z})^{10} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{10} + \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{10}$$

$$= \left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) + \left(\cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2}\right)$$

$$= 2\cos\frac{5\pi}{2}$$

$$= 2(0)$$

$$= 0.$$

$$= RHS$$

$\left[\cos\frac{n\pi}{2} = 0 \text{ if } n \text{ is odd.}\right.$

Q.5 **(ii)** $\left(1 + i\sqrt{3}\right)^n + \left(1 - i\sqrt{3}\right)^n = 2^{n+1}cos(n\pi/3)$. prove this

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$1 - i\sqrt{3} = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^n\left\{\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right\}$$

$$= 2^n\left\{2\cos\frac{n\pi}{3}\right\}$$

$$= 2^{n+1}\cos\frac{n\pi}{3}$$

6. If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2.2^{n/2}\cos n\pi/4$, Hence, deduce that $\alpha^8 + \beta^8 = 32$

Soln :-

$$x^2 - 2x + 2 = 0$$

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{}$$

$$2a$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

let $\alpha = 1+i$, $\beta = 1-i$

$$\alpha = 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\beta = 1-i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$\alpha^n + \beta^n = (\sqrt{2})^n\left\{\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right\}$$

$$= (\sqrt{2})^n\left\{2\cos\frac{n\pi}{4}\right\}$$

$$= 2\left(2^{n/2}\right)\cos\left(\frac{n\pi}{4}\right) = RHS$$

Now put $n=8$

$$\alpha^8 + \beta^8 = 2\left(2^{8/2}\right)\cos\left(\frac{8\pi}{4}\right) = 2(2^4)\cos(2\pi)$$

$$= 2^5 = 32$$

7. If α, β are the roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$, Prove that $\alpha^3 + \beta^3 = 0$ and $\alpha^3 - \beta^3 = 16\,i$ (HW)

8. If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$, $c = \cos 2\gamma + i\sin 2\gamma$, prove that $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2\cos(\alpha + \beta - \gamma)$

Soln :-

$$a = \cos 2\alpha + i\sin 2\alpha = e^{i2\alpha}$$

$$b = \cos 2\beta + i\sin 2\beta = e^{i2\beta}$$

$$c = \cos 2\gamma + i\sin 2\gamma = e^{i2\gamma}$$

$$\frac{ab}{c} = \frac{e^{i2\alpha}\cdot e^{i2\beta}}{e^{i2\gamma}} = e^{i2(\alpha+\beta-\gamma)}$$

also $\frac{c}{ab} = e^{-i2(\alpha+\beta-\gamma)}$

also $\frac{c}{ab} = e^{-i2(\alpha+\beta-\gamma)}$

Now $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = \sqrt{e^{i2(\alpha+\beta-\gamma)}} + \sqrt{e^{-i2(\alpha+\beta-\gamma)}}$

$= e^{i(\alpha+\beta-\gamma)} + e^{-i(\alpha+\beta-\gamma)}$

$= \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)$
$+ \cos(\alpha+\beta-\gamma) - i\sin(\alpha+\beta-\gamma)$

$= 2\cos(\alpha+\beta-\gamma)$

9. If $x - \frac{1}{x} = 2i\sin\theta$, $y - \frac{1}{y} = 2\,i\sin\Phi$, $z - \frac{1}{z} = 2i\sin\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta+\Phi+\psi)$ **(ii)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\,cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Sol^n:- we have $x - \frac{1}{x} = 2i\sin\theta$

$x^2 - 1 = 2i\sin\theta\, x$

$x^2 - 2i\sin\theta\, x - 1 = 0$

This is a quadratic in x

$ax^2 + bx + c = 0$

$a = 1,\ b = -2i\sin\theta,\ c = -1$

$\therefore$ roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{2i\sin\theta \pm \sqrt{-4\sin^2\theta + 4}}{2}$

$= \frac{2i\sin\theta \pm 2\sqrt{1-\sin^2\theta}}{2}$

$$x = i\sin\theta \pm \cos\theta$$

Let $x = \cos\theta + i\sin\theta = e^{i\theta}$

similarly $y = \cos\phi + i\sin\phi = e^{i\phi}$

$z = \cos\psi + i\sin\psi = e^{i\psi}$

Now $xyz = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi)$

$= \cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)$

Now $\frac{1}{xyz} = \cos(\theta+\phi+\psi) - i\sin(\theta+\phi+\psi)$

$\therefore xyz + \frac{1}{xyz} = 2\cos(\theta+\phi+\psi)$

(ii) $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}}$

$$\frac{\sqrt[m]{x}}{\sqrt[n]{y}} = \frac{(x)^{1/m}}{(y)^{1/n}} = \frac{(\cos\theta + i\sin\theta)^{\frac{1}{m}}}{(\cos\phi + i\sin\phi)^{\frac{1}{n}}} = \frac{\cos\left(\frac{\theta}{m}\right) + i\sin\left(\frac{\theta}{m}\right)}{\cos\left(\frac{\phi}{n}\right) + i\sin\left(\frac{\phi}{n}\right)}$$

$$= \left[\cos\left(\frac{\theta}{m}\right) + i\sin\left(\frac{\theta}{m}\right)\right]\left[\cos\left(\frac{\phi}{n}\right) - i\sin\left(\frac{\phi}{n}\right)\right]$$

$$\frac{\sqrt[m]{x}}{\sqrt[n]{y}} = \cos\left(\frac{\theta}{m} - \frac{\phi}{n}\right) + i\sin\left(\frac{\theta}{m} - \frac{\phi}{n}\right)$$

similarly $\frac{\sqrt[n]{y}}{\sqrt[m]{x}} = \cos\left(\frac{\theta}{m} - \frac{\phi}{n}\right) - i\sin\left(\frac{\theta}{m} - \frac{\phi}{n}\right)$

$$\therefore \frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\cos\left(\frac{\theta}{m} - \frac{\phi}{n}\right)$$

10. If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, Prove that $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$.

Soln :- $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$

$\therefore (\cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$

$(\cos\alpha + i\sin\alpha) + 2(\cos\beta + i\sin\beta) + 3(\cos\gamma + i\sin\gamma) = 0$

$$\text{Let}\quad \left.\begin{aligned} x &= \cos\alpha + i\sin\alpha \\ y &= 2(\cos\beta + i\sin\beta) \\ z &= 3(\cos\gamma + i\sin\gamma) \end{aligned}\right\} \rightarrow x+y+z=0$$

$\therefore (x+y+z)^3 = 0$

$(x^3+y^3+z^3) + 3\underbrace{(x+y+z)}_{0}(xy+yz+zx) - 3xyz = 0.$

$x^3+y^3+z^3 = 3xyz$

$(\cos\alpha + i\sin\alpha)^3 + [2(\cos\beta + i\sin\beta)]^3 + [3(\cos\gamma + i\sin\gamma)]^3$

$= 3(\cos\alpha + i\sin\alpha)\cdot 2(\cos\beta + i\sin\beta)\cdot 3(\cos\gamma + i\sin\gamma)$

$\Rightarrow (\cos 3\alpha + i\sin 3\alpha) + 8(\cos 3\beta + i\sin 3\beta) + 27(\cos 3\gamma + i\sin 3\gamma)$

$= 18[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$

$\Rightarrow (\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma) + i(\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma)$

$= 18\cos(\alpha+\beta+\gamma) + i\,18\sin(\alpha+\beta+\gamma)$

comparing the imaginary part, we get the answer.

11.

If $x_r = cos\frac{\pi}{3^r} + isin\frac{\pi}{3^r}$, prove that **(i)** $x_1x_2x_3 \ldots ad.inf. = i$ **(ii)** $x_0x_1x_2 \ldots ad.inf. = -i$

Soln :- $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$

$\therefore x_0 = \cos\frac{\pi}{3^0} + i\sin\frac{\pi}{3^0} = \cos\pi + i\sin\pi = -1$

$$x_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$x_2 = \cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}$$

$$x_3 = \cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}$$

(i) $x_1 x_2 x_3 \cdots$ ad inf

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}\right)\cdots$$

$$= \cos\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots\right)$$

but $\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots$ is infinite sum of G.P.

where $a = \frac{\pi}{3}$ $\quad r = \frac{1}{3}$

$$\therefore \frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots = \frac{a}{1-r} = \frac{\pi/3}{1-1/3} = \frac{\pi}{2}$$

$$\therefore \text{LHS} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = 0 + i(1) = i$$

(ii) $x_0 x_1 x_2 x_3 \cdots$ ad inf

$= x_0 (x_1 x_2 x_3 \cdots$ ad inf$)$

$= x_0 (i)$ $\quad$ (from first part)

$= (-1)(i)$

$= -i = \text{RHS}$

12. If $(\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)\ldots\ldots[\cos(2n-1)\theta + i\sin(2n-1)\theta] = 1$ then show that the general value of θ is $\frac{2r\pi}{n^2}$

Sol^n :- $(\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)\cdots[\cos(2n-1)\theta + i\sin(2n-1)\theta] = 1$

$$\cos(\theta + 3\theta + 5\theta + \cdots (2n-1)\theta) + i\sin(\theta + 3\theta + 5\theta + \cdots + (2n-1)\theta) = 1$$

$\cos(\theta + 3\theta + 5\theta + \cdots (2n-1)\theta) + i\sin(\theta + 3\theta + 5\theta + \cdots + (2n-1)\theta) = 1$

$\cos(1+3+5+\cdots(2n-1))\theta + i\sin(1+3+5+\cdots+(2n-1))\theta = 1$

$1+3+5+\cdots+(2n-1)$ is an A.P. with first term 1, the number of terms n and common difference $= 2$

The sum $S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[2+(n-1)2]$

$= n^2$

$\therefore \cos(n^2\theta) + i\sin(n^2\theta) = 1$

$\cos(n^2\theta) + i\sin(n^2\theta) = \cos 0 + i\sin 0$ → principal value

$= \cos(2r\pi) + i\sin(2r\pi)$ → general

comparing both sides

$n^2\theta = 2r\pi$

$\therefore \boxed{\theta = \frac{2r\pi}{n^2}}$

13. By using De Moivre's Theorem show that $\sin\alpha + \sin 2\alpha + \cdots + \sin 5\alpha = \frac{\sin 3\alpha \sin(5\alpha/2)}{\sin\alpha/2}$

$S_n = \frac{a - r^n}{a - r}$

$a = 1, r = z$

Sol^n :- $\frac{1-z^6}{1-z} = 1 + z + z^2 + z^3 + z^4 + z^5$ —— (i)

Let $z = \cos\alpha + i\sin\alpha$ then by De-Moivre's thm

$z^n = \cos n\alpha + i\sin n\alpha$

$1 + z + z^2 + z^3 + z^4 + z^5 = 1 + (\cos\alpha + i\sin\alpha) + (\cos\alpha + i\sin\alpha)^2 + (\cos\alpha + i\sin\alpha)^3 + \cdots + (\cos\alpha + i\sin\alpha)^5$

$= 1 + (\cos\alpha + i\sin\alpha) + (\cos 2\alpha + i\sin 2\alpha) + (\cos 3\alpha + i\sin 3\alpha) + (\cos 4\alpha + i\sin 4\alpha) + (\cos 5\alpha + i\sin 5\alpha)$

$= (1 + \cos\alpha + \cos 2\alpha + \cos 3\alpha + \cos 4\alpha + \cos 5\alpha)$

$$= (1+\cos\alpha+\cos 2\alpha+\cos 3\alpha+\cos 4\alpha+\cos 5\alpha)$$
$$+ i(\sin\alpha+\sin 2\alpha+\sin 3\alpha+\sin 4\alpha+\sin 5\alpha) \quad \text{— (ii)}$$

Now $$\frac{1-z^6}{1-z} = \frac{1-(\cos\alpha+i\sin\alpha)^6}{1-(\cos\alpha+i\sin\alpha)} = \frac{1-(\cos 6\alpha+i\sin 6\alpha)}{1-(\cos\alpha+i\sin\alpha)}$$

$$= \frac{(1-\cos 6\alpha)-i\sin 6\alpha}{(1-\cos\alpha)-i\sin\alpha}$$

$$= \frac{2\sin^2 3\alpha - 2i\sin 3\alpha\cos 3\alpha}{2\sin^2(\alpha/2) - 2i\sin\alpha/2\cos\alpha/2}$$

$$= \frac{2\sin 3\alpha}{2\sin(\alpha/2)} \frac{[\sin 3\alpha - i\cos 3\alpha]}{[\sin\alpha/2 - i\cos\alpha/2]} \times \frac{[\sin\alpha/2+i\cos\alpha/2]}{[\sin\alpha/2+i\cos\alpha/2]}$$

$$= \frac{\sin 3\alpha}{\sin(\alpha/2)} \frac{[\sin 3\alpha - i\cos 3\alpha][\sin\alpha/2+i\cos\alpha/2]}{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}}$$

$$= \frac{\sin 3\alpha}{\sin(\alpha/2)} \left[\cos\left(\frac{\pi}{2}-3\alpha\right) - i\sin\left(\frac{\pi}{2}-3\alpha\right)\right]\left[\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right]$$

$$= \frac{\sin 3\alpha}{\sin(\alpha/2)} \times e^{-i(\frac{\pi}{2}-3\alpha)} \cdot e^{i(\frac{\pi}{2}-\frac{\alpha}{2})}$$

$$= \frac{\sin 3\alpha}{\sin(\alpha/2)} e^{i(-\frac{\pi}{2}+3\alpha+\frac{\pi}{2}-\frac{\alpha}{2})}$$

$$= \frac{\sin 3\alpha}{\sin(\alpha/2)} e^{i(\frac{5\alpha}{2})}$$

$$\frac{1-z^6}{1-z} = \frac{\sin 3\alpha}{\sin(\alpha/2)}\left[\cos\left(\frac{5\alpha}{2}\right)+i\sin\left(\frac{5\alpha}{2}\right)\right] \quad \text{— (iii)}$$

from (i), (ii) and (iii), comparing the imaginary parts

$$\sin\alpha + \sin 2\alpha + \sin 3\alpha + \sin 4\alpha + \sin 5\alpha = \frac{\sin 3\alpha}{\sin(\alpha/2)} \cdot \sin\left(\frac{5\alpha}{2}\right)$$

Applications of De-Moivre's Theorem

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no. of roots = deg of eqn

$z^4 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $n=4$

$z^4 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= \cos\left(2k\pi + \frac{\pi}{3}\right)$

ROOTS OF ALGEBRAIC EQUATIONS:

De Moivre's theorem can be used to find the roots of an algebraic equation.
General values of $\cos\theta = \cos(2k\pi + \theta)$ and $\sin\theta = \sin(2k\pi + \theta)$ where k is an integer.

To solve the equation of the type $z^n = \cos\theta + i\sin\theta$, we apply De Moivre's theorem

$$z = (\cos\theta + i\sin\theta)^{\frac{1}{n}} = \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$$

This shows that $\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ is one of the n roots of $z^n = \cos\theta + i\sin\theta$.

The other roots are obtain by expressing the number in the general form

$$z = \{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\}^{\frac{1}{n}} = \cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)$$

Taking k = 0, 1, 2,..............,(n – 1). We get n roots of the equation.

Note: (i) Complex roots always occur in conjugate pair if coefficients of different powers of x including constant terms in the equation are real.

(ii) Continued products mean products of all the roots of the equation.

SOME SOLVED EXAMPLES:

1. If ω is a cube root of unity, prove that $(1-\omega)^6 = -27$

Soln: let $z^3 = 1$ $\therefore z = (1)^{1/3}$

$\therefore z = (\cos 0 + i\sin 0)^{1/3} = (\cos 2k\pi + i\sin 2k\pi)^{1/3}$

$= \left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right)$ where $k = 0, 1, 2$

$k=0$ $\therefore z_0 = \cos 0 + i\sin 0 = 1$

$k=1$, $z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$

$k=2$, $z_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \omega^2$

Now $1 + \omega + \omega^2 = 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$$= 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - 1 = 0$$

$\therefore \; 1 + \omega^2 = -\omega$

$$(1-\omega)^6 = \left((1-\omega)^2\right)^3 = \left(1 - 2\omega + \omega^2\right)^3 = \left(1 + \omega^2 - 2\omega\right)^3$$

$$= (-\omega - 2\omega)^3 = (-3\omega)^3 = -27\omega^3$$

but $\omega^3 = 1$

$$\therefore \; (1-\omega)^6 = -27.$$

2. Find all the values of $\sqrt[3]{(1+i)/\sqrt{2}} + \sqrt[3]{(1-i)/\sqrt{2}}$

n^{th} root $\rightarrow$ n values

soln: let $z = \sqrt[3]{(1+i)/\sqrt{2}} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{1/3}$

$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3}$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{1/3}$$

$$= \left[\cos\left(\frac{8k+1}{4}\right)\pi + i\sin\left(\frac{8k+1}{4}\right)\right]^{1/3}$$

$$\sqrt[3]{(1+i)/\sqrt{2}} = \cos\left(\frac{8k+1}{12}\right)\pi + i\sin\left(\frac{8k+1}{12}\right)\pi$$

where $k = 0, 1, 2$

similarly $\sqrt[3]{(1-i)/\sqrt{2}} = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^{1/3}$

$$= \cos\left(\frac{8k+1}{12}\right)\pi - i\sin\left(\frac{8k+1}{12}\right)\pi$$

$$= \cos\left(\frac{8k+1}{12}\right)\pi - i\sin\left(\frac{8k+1}{12}\right)\pi$$

$$k = 0, 1, 2$$

$$\therefore \sqrt[3]{(1+i)/\sqrt{2}} + \sqrt[3]{(1-i)/\sqrt{2}} = 2\cos\left(\frac{8k+1}{12}\right)\pi \quad \text{where } k=0,1,2$$

$$= 2\cos\frac{\pi}{12},\ 2\cos\frac{9\pi}{12},\ 2\cos\frac{17\pi}{12}$$

3. Find the cube roots of $(1 - \cos\theta - i\sin\theta)$. $\underline{n=3}$

Soln :- $(1-\cos\theta - i\sin\theta)^{1/3}$

$$= \left(2\sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{1/3}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{1/3}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]^{1/3}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{1/3}\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(\frac{\theta}{2}-\frac{\pi}{2}\right) + i\sin\left(\frac{\theta}{2}-\frac{\pi}{2}\right)\right]^{1/3}$$

$$= \left[2\sin\left(\frac{\theta}{2}\right)\right]^{1/3}\left[\cos\left(2k\pi + \frac{\theta}{2} - \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\theta}{2} - \frac{\pi}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{1/3}\left[\cos\left(\frac{(4k-1)\pi+\theta}{2}\right) + i\sin\left(\frac{(4k-1)\pi+\theta}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\frac{\theta}{2}\right)^{1/3}\left[\cos\left(\frac{(4k-1)\pi+\theta}{6}\right) + i\sin\left(\frac{(4k-1)\pi+\theta}{6}\right)\right]$$

$$= \left(2\sin\frac{\alpha}{2}\right) \left[\cos\left(\frac{\quad}{6}\right) + i\sin\left(\frac{\quad}{6}\right)\right]$$

Putting $k = 0, 1, 2$ we get all the roots.

4. Find the continued product of all the value of $(-i)^{2/3}$

$\frac{2}{3} = \frac{2 \to \text{Power}}{3 \to \text{root}}$

$n = 3$

Soln:- $(-i)^{2/3} = \left[(-i)^2\right]^{1/3} = (-1)^{1/3}$

$$= \left[\cos\pi + i\sin\pi\right]^{1/3}$$

$$= \left[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\right]^{1/3}$$

$$= \cos\left(\frac{2k+1}{3}\right)\pi + i\sin\left(\frac{2k+1}{3}\right)\pi$$

where $k = 0, 1, 2$

$\therefore$ roots are $z_0 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ for $k = 0$

$z_1 = \cos\pi + i\sin\pi$ for $k = 1$

$z_2 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$ for $k = 2$

$\therefore$ The continued product $= z_0 \cdot z_1 \cdot z_2$

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\pi + i\sin\pi\right)\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

$$= \cos\left(\frac{\pi}{3} + \pi + \frac{5\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \pi + \frac{5\pi}{3}\right)$$

$$= \cos(3\pi) + i\sin(3\pi)$$

$$= (-1) + i(0) = -1$$

5. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1. (H.W.)

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4} = \left[\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3\right]^{1/4}$$

5. Find all the values of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1. (H.W.)

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{3/4} = \left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{3}\right]^{1/4}$$

$$= \left(\cos\pi + i\sin\pi\right)^{1/4}$$

6. SOLVE: $x^7+x^4+x^3+1=0$

Soln :- $x^7+x^4+x^3+1=0$

$$x^4(x^3+1)+(x^3+1)=0$$

$$(x^3+1)(x^4+1)=0$$

Now $x^3+1=0 \Rightarrow x^3=-1 \Rightarrow x^3=\cos\pi+i\sin\pi$

$$\Rightarrow x^3 = \cos(2k\pi+\pi)+i\sin(2k\pi+\pi)$$

$$x^3 = \cos(2k+1)\pi + i\sin(2k+1)\pi$$

$$\Rightarrow x = \left[\cos(2k+1)\pi + i\sin(2k+1)\pi\right]^{1/3}$$

$$x = \cos\left(\frac{2k+1}{3}\right)\pi + i\sin\left(\frac{2k+1}{3}\right)\pi$$

where $k=0,1,2$

$\therefore$ The roots are

$\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $\cos\pi+i\sin\pi$, $\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$

Now :- $x^4+1=0 \Rightarrow x^4=-1 \Rightarrow x^4 = \cos\pi+i\sin\pi$

$$= \cos(2k+1)\pi + i\sin(2k+1)\pi$$

$$\therefore x = \cos\left(\frac{2k+1}{4}\right)\pi + i\sin\left(\frac{2k+1}{4}\right)\pi \qquad k=0,1,2,3$$

$\therefore$ The next 4 roots are

$\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$, $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$, $\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

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7. SOLVE: $x^4 + x^3 + x^2 + x + 1 = 0$

$x^4 + x^3 + x^2 + x + 1 = 0$

multiply by $x - 1$

$x^5 - 1 = 0$

$x^5 = 1 \Rightarrow \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$

$x = (\cos 2k\pi + i\sin 2k\pi)^{1/5}$

$= \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$ where $k = 0, 1, 2, 3, 4$

$x_0 = \cos 0 + i\sin 0 = 1$

$x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, $x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$, $x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

Here $x_0 = 1$ is root of $x - 1 = 0$ and x_1, x_2, x_3, x_4 are the roots of given eqn.

Ex:- H.W

$x^4 + x^2 + 1 = 0$ or $x^6 + x^3 + 1 = 0$

multiply $(x^2 - 1)$ — multi by $(x^3 - 1)$

$x^6 - 1 = 0$ — $x^9 - 1 = 0$

$\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}$

$k = 0, \ldots 8$

8. SOLVE: $x^4 - x^2 + 1 = 0$

multiply by $x^2 + 1$

$x^6 + 1 = 0$

$$x^6 = -1 = \cos\pi + i\sin\pi = \cos(2k+1)\pi + i\sin(2k+1)\pi$$

$$x = \left[\cos(2k+1)\pi + i\sin(2k+1)\pi\right]^{1/6}$$

$$= \cos\frac{(2k+1)\pi}{6} + i\sin\frac{(2k+1)\pi}{6}$$

where $k = 0, 1, 2, \cdots, 5$

$x_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{2}$

$x_1 = \cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = i$

$x_2 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$x_3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$x_4 = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i(-1) = -i$

$x_5 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

x_1, x_4 are the roots of $x^2+1=0$ and x_0, x_2, x_3, x_5 are the roots of given eqn $x^4 - x^2 + 1 = 0$.

Ex: $x^4 - x^3 + x^2 - x + 1 = 0$.

multiply by $x+1 \Rightarrow x^5 + 1 = 0$.

9. Find the roots common to $x^4 + 1 = 0$ *and* $x^6 - i = 0$. (H.W.)

$x^4+1=0$

$x=(-1)^{1/4}$

$= \cos\frac{(2K+1)\pi}{4} + i\sin\frac{(2K+1)\pi}{4}$

$K=0,1,2,3$

$x_0 =$

$x_1 =$

$x_2 =$

$x_3 =$

$x^6 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= \cos\left(\frac{4K+1}{2}\right)\pi + i\sin\left(\frac{4K+1}{2}\right)\pi$

$x = \cos\left(\frac{4K+1}{12}\right)\pi + i\sin\left(\frac{4K+1}{12}\right)\pi$

$k=0,1,2,\dots,5$

$x_0 =$

$x_1 =$

$x_2 =$

$x_3 =$

$x_4 =$

$x_5 =$

10. If $(1+x)^6+x^6=0$ show that $x=-\frac{1}{2}-\frac{i}{2}\cot\frac{\theta}{2}$ where $\theta=(2n+1)\pi/6, n=0,1,2,3,4,5.$

Soln:- $(1+x)^6+x^6=0$

$\left(\frac{1+x}{x}\right)^6+1=0$

$\left(\frac{1+x}{x}\right)^6=-1=\cos\pi+i\sin\pi=\cos(2n+1)\pi+i\sin(2n+1)\pi$

$\frac{1+x}{x}=\left[\cos(2n+1)\pi+i\sin(2n+1)\pi\right]^{1/6}$

$=\cos\left(\frac{2n+1}{6}\right)\pi+i\sin\left(\frac{2n+1}{6}\right)\pi$ where $n=0,1,2,3,4,5.$

Let $(2n+1)\pi = \theta$

$\left(\frac{\quad}{6}\right)\pi = 0$

$\therefore \frac{1+x}{x} = \cos\theta + i\sin\theta$

$\therefore \frac{1}{x} + 1 = \cos\theta + i\sin\theta$

$\frac{1}{x} = (\cos\theta - 1) + i\sin\theta$

$$\therefore x = \frac{1}{(\cos\theta - 1) + i\sin\theta}$$

$$= \frac{1}{(\cos\theta-1)+i\sin\theta} \times \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)-i\sin\theta}$$

$$= \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)^2+\sin^2\theta} = \frac{(\cos\theta-1)-i\sin\theta}{2(1-\cos\theta)}$$

$$= -\frac{1}{2} - \frac{i}{2}\,\frac{\sin\theta}{1-\cos\theta}$$

$$= -\frac{1}{2} - \frac{i}{2}\,\frac{2\sin\theta/2\,\cos\theta/2}{2\sin^2\theta/2}$$

$$x = -\frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2} \quad \text{where } \theta = \left(\frac{2n+1}{6}\right)\pi$$

11. If one root of $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0\ is\ 1 + i,$ find all other roots.

Soln:- $1+i$ is a root of $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$

$\therefore$ $1-i$ is also a root of $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$

(complex roots always occur in pairs)

Now $x = 1 \pm i$ are the two roots

$x - 1 = \pm i$

$(x-1)^2 = -1$

$x^2 - 2x + 1 = -1 \Rightarrow x^2 - 2x + 2 = 0$

Now to find the remaining roots, we will divide the given eqn by $x^2 - 2x + 2$

$(x^4 - 6x^3 + 15x^2 - 18x + 10) = (x^2 - 2x + 2)(x^2 - 4x + 5)$

$\therefore$ the remaining two roots are the roots of equation $x^2 - 4x + 5 = 0$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{4^2 - 4(1)(5)}}{2(1)}$$

$x = 2 \pm i$

$\therefore$ the required remaining roots are $1-i$ and $2 \pm i$

12. If $\alpha, \alpha^2, \alpha^3, \alpha^4$, are the roots of $x^5 - 1 = 0$, find them & show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$.

$x^5 - 1 = 0$

$x^5 = 1 = \cos 2k\pi + i \sin 2k\pi$

$x = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}$

$k = 0, 1, 2, 3, 4$

$x_0 = \cos 0 + i \sin 0 = 1$

$$x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$$

$$x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = \alpha^2$$

$$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \alpha^3$$

$$x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \alpha^4$$

$1, \alpha, \alpha^2, \alpha^3$ and α^4 are the roots of $x^5 - 1 = 0$

$$\therefore x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\frac{x^5-1}{x-1} = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$x^4 + x^3 + x^2 + x + 1 = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

put $x = 1$

$$1+1+1+1+1 = (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)$$

$$\therefore (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$$

13. Solve the equation $z^4 = i(z-1)^4$ and show that the real part of all the roots is $1/2$. (H.W.)

$$\left(\frac{z}{z-1}\right)^4 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= \cos\left(\frac{4k+1}{2}\right)\pi + i\sin\left(\frac{4k+1}{2}\right)\pi$$

$$\frac{z}{z-1} = \cos\frac{(4k+1)}{8}\pi + i\sin\left(\frac{4k+1}{8}\right)\pi$$

$(4k+1)\ldots$

$$\left(\frac{}{8} \right) \parallel = \theta$$

$$\frac{z}{z-1} = \cos\theta + i\sin\theta$$

14. If ω is a 7th root of unity, prove that $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} = 7$ if n is a multiple of 7 and is equal to zero otherwise.

Soln :- $x = (1)^{1/7} = (\cos 2k\pi + i\sin 2k\pi)^{1/7}$

$$= \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7} \quad k = 0, 1, 2, 3, 4, 5, 6$$

Let $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$\therefore \omega^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7 = \cos 2\pi + i\sin 2\pi = 1.$

$\therefore \omega^{7n} = (\omega^7)^n = (1)^n = 1.$ $\quad \boxed{\omega^{7n} = 1}$

If n is not a multiple of 7 then $\omega^n \neq 1$

Here $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n}$

when n is a multiple of 7 ie $n = 7k$

$\therefore S = 1 + \omega^{7k} + \omega^{2(7k)} + \omega^{3(7k)} + \cdots + \omega^{6(7k)}$

$= 1 + (\omega^7)^k + (\omega^7)^{2k} + (\omega^7)^{3k} + \cdots + (\omega^7)^{6k}$

$= 1 + (1)^k + (1)^{2k} + (1)^{3k} + \cdots + (1)^{6k}$

$= 1 + 1 + 1 + 1 + 1 + 1 + 1$

$\therefore S = 7.$

If n is not a multiple of 7

$\omega^n \neq 1.$

$$S = 1 + \omega^{n} + \omega^{2n} + \omega^{3n} + \cdots + \omega^{6n}$$

$$= \frac{1-\omega^{7n}}{1-\omega^{n}} \quad \left(\text{sum of 7 terms of G.P. } a=1,\ r=\omega^{n}\right)$$

Now $\omega^{7n} = 1$ & $\omega^{n} \neq 1$

$$\therefore S = \frac{1-1}{1-\omega^{n}} = \frac{0}{1-\omega^{n}} = 0.$$

15. Prove that $\sqrt{1+sec(\theta/2)} = \left(1+e^{i\theta}\right)^{-1/2} + \left(1+e^{-i\theta}\right)^{-1/2}$

Soln :- To show that $\sqrt{1+\sec\left(\frac{\theta}{2}\right)} = \frac{1}{\sqrt{1+e^{i\theta}}} + \frac{1}{\sqrt{1+e^{-i\theta}}}$

squaring both sides

$$1+\sec\frac{\theta}{2} = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$$

we will prove this result.

$$RHS = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$$

$$= \frac{1}{1+e^{i\theta}} + \frac{e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{1+e^{i\theta}+e^{-i\theta}+1}}$$

$$= \frac{1+e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{2+(e^{i\theta}+e^{-i\theta})}}$$

Now $e^{i\theta}+e^{-i\theta} = 2\cos\theta$

$$= 1 + \frac{2}{\sqrt{2+2\cos\theta}} = 1 + \frac{2}{\sqrt{2(1+\cos\theta)}}$$

$\sqrt{2+2\cos\theta}$ $\sqrt{2(1+\cos\theta)}$

$$= 1 + \frac{2}{\sqrt{2\left(2\cos^2\frac{\theta}{2}\right)}} = 1 + \frac{2}{2\cos\theta/2}$$

$$= 1 + \sec\frac{\theta}{2}$$

$= LHS.$

HYPERBOLIC FUNCTIONS

Monday, October 25, 2021 1:00 PM

CIRCULAR FUNCTIONS:

From Euler's formula, we have $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore\ \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

If $z = x + iy$ is complex number, then $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$

These are called circular function of complex numbers.

HYPERBOLIC FUNCTIONS:

If x is real or complex, then sine hyperbolic of x is denoted by $sinh\ x$ and is given as, $\sinh x = \frac{e^{x}-e^{-x}}{2}$ and

Cosine hyperbolic of x is denoted by $cosh\ x$ and is given as, $\cosh x = \frac{e^{x}+e^{-x}}{2}$

From above expressions, other hyperbolic functions can also be obtained as

$\tan hx = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}, \qquad \mathrm{cosech}x = \frac{1}{\sinh x} = \frac{2}{e^{x}-e^{-x}}, \qquad \mathrm{sech}\,x = \frac{1}{\cosh x} = \frac{2}{e^{x}+e^{-x}}$, and

$\coth x = \frac{1}{\tanh x} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

TABLE OF VALUES OF HYPERBOLIC FUNCTION:

From the definitions of sinhx, cos x, tanhx, we can obtain the following values of hyperbolic function.

x	$-\infty$	0	∞
$\sinh x$	$-\infty$	0	∞
$\cosh x$	∞	1	∞
$\tanh x$	-1	0	1

Note: since $\tanh(-\infty) = -1, \tanh(0) = 0, \tanh(\infty) = 1$

$\therefore |\tanh x| \le 1$

RELATION BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS :

(i)	$\sin ix = i\sinh x$ & $\sinh x = -i\sin ix$	$\sinh ix = i\sin x$ & $\sin x = -i\sinh ix$
(ii)	$\cos ix = \cosh x$	$\cosh ix = \cos x$
(iii)	$\tan ix = i\tanh x$ & $\tanh x = -i\tan ix$	$\tanh ix = i\tan x$ & $\tan x = -i\tanh ix$

FORMULAE ON HYPERBOLIC FUNCTIONS :

	CIRCULAR FUNCTIONS	HYPERBOLIC FUNCTIONS
1	$\sin(-x) = -(\sin x)$	$\sinh(-x) = -\sinh x,$
2	$\cos(-x) = (cosx)$	$\cosh(-x) = \cosh x$
3	$e^{ix} = \cos x + i\sin x$	$e^{x} = \cosh x + \sinh x$
4	$e^{-ix} = \cos x - i\sin x$	$e^{-x} = \cosh x - \sinh x$
5	$sin^2x + cos^2x = 1$	$\cosh^2x - \sinh^2x = 1$
6	$1 + \tan^2x = \sec^2x$	$\mathrm{sech}^2x + \tanh^2x = 1$
7	$1 + \cot^2x = \mathrm{cosec}^2x$	$\coth^2x - \mathrm{cosech}^2x = 1$
8	$\sin 2x = 2\sin x\cos x$ $= \frac{2\tan x}{1+\tan^2 x}$	$\sinh 2x = 2\sinh x\cosh x$ $= \frac{2\tanh x}{1-\tanh^2 x}$
9	$\cos 2x = \cos^2x - \sin^2x$ $= 2\cos^2x - 1$	$\cosh 2x = \cosh^2x + \sinh^2x$ $= 2\cosh^2x - 1$

	$= \frac{}{1+\tan^2 x}$	$= \frac{}{1-\tanh^2 x}$
9	$\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $= \frac{1-\tan^2 x}{1+\tan^2 x}$	$\cosh 2x = \cosh^2 x + \sinh^2 x$ $= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $= \frac{1+\tanh^2 x}{1-\tanh^2 x}$
10	$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$	$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$
11	$\sin 3x = 3\sin x - 4\sin^3 x$	$\sinh 3x = 3\sinh x + 4\sinh^3 x$
12	$\cos 3x = 4\cos^3 x - 3\cos x$	$\cosh 3x = 4\cosh^3 x - 3\cosh x$
13	$\tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}$	$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$
14	$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
15	$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
16	$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tanh y}$	$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
17	$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$	$\coth(x \pm y) = \frac{-\coth x \coth y \mp 1}{\coth y \pm \coth x}$
18	$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$	$\sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$
19	$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$	$\sinh x - \sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}$
20	$\cos x + \cos y$ $= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$	$\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$
21	$\cos x - \cos y$ $= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$	$\cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$
22	$2\sin x\cos y = \sin(x+y) + \sin(x-y)$	$2\sinh x\cosh y = \sinh(x+y) + \sinh(x-y)$
23	$2\cos x\sin y = \sin(x+y) - \sin(x-y)$	$2\cosh x\sinh y = \sinh(x+y) - \sinh(x-y)$
24	$2\cos x\cos y = \cos(x+y) + \cos(x-y)$	$2\cosh x\cosh y = \cosh(x+y) + \cosh(x-y)$
25	$2\sin x\sin y = \cos(x-y) - \cos(x+y)$	$2\sinh x\sinh y = \cos h(x+y) - \cos h(x-y)$

PERIOD OF HYPERBOLIC FUNTIONS:

$\sinh(2\pi i + x) = \sinh(2\pi i)\cosh x + \cosh(2\pi i)\sinh x$

$= i\sin 2\pi \cosh x + \cos 2\pi \sinh x$

$= 0 + \sinh x \qquad = \sinh x$

Hence $sinh\ x$ is a periodic function of period $2\pi i$

Similarly we can prove that $cosh\ x$ and $tanh\ x$ are periodic functions of period $2\pi i$ and πi .

DIFFERENTIATION AND INTRGRATION :

(i) If $y = \sinh x$,

$$y = \frac{e^x - e^{-x}}{2}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

If $y = \sinh x$, $\frac{dy}{dx} = \cosh x$

(ii) If $y = cosh\ x$,

$$y = \frac{e^x + e^{-x}}{2},$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

If $y = cosh\ x$, $\frac{dy}{dx} = \sinh x$

(iii) If $y = \tanh x$,

$$y = \frac{\sinh x}{\cosh x}$$

$$\therefore \frac{dy}{dx} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{cosh^2 x} = \frac{1}{cosh^2 x} = sech^2 x$$

If $y = \tanh x$, $\frac{dy}{dx} = sech^2 x$

Hence, we get the following three results

$\int \cosh x\, dx = \sinh x$, $\int \sinh x\, dx = \cosh x$, $\int sech^2 x dx = \tanh x$

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SOME SOLVED EXAMPLES:

Soln: 1^{st} method :-

1. If tanh x $= \frac{1}{2}$, $find\ \sinh 2x\ and\ \cosh 2x$

$$\sinh 2x = \frac{2\tanh x}{1 - \tanh^2 x} = \frac{2 \cdot \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

$$\cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x} = \frac{1 + \left(\frac{1}{2}\right)^2}{1 - \left(\frac{1}{2}\right)^2} = \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{3}$$

2^{nd} method

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}, \quad \cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$$

given $\tanh x = \frac{1}{2}$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{2} \Rightarrow \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1}{2} \Rightarrow 2e^{2x} - 2 = e^{2x} + 1$$

$$\Rightarrow e^{2x} = 3 \Rightarrow e^{-2x} = \frac{1}{3}$$

$$\Rightarrow e^{2x} = 3 \Rightarrow e^{-2x} = \frac{1}{3}$$

$$\therefore \sinh 2x = \frac{3 - \frac{1}{3}}{2} = \frac{8}{6} = \frac{4}{3}$$

$$\cosh 2x = \frac{3 + \frac{1}{3}}{2} = \frac{10}{6} = \frac{5}{3}$$

2. Solve the equation $7\cosh x + 8\sinh x = 1$ for real values of x.

Soln :- $7\cosh x + 8\sinh x = 1$

$$7\left(\frac{e^x + e^{-x}}{2}\right) + 8\left(\frac{e^x - e^{-x}}{2}\right) = 1$$

$$7e^x + 7e^{-x} + 8e^x - 8e^{-x} = 2$$

$$15e^x - e^{-x} = 2$$

$$15e^{2x} - 1 = 2e^x \Rightarrow 15e^{2x} - 2e^x - 1 = 0$$

This is a quadratic in e^x $\quad 15y^2 - 2y - 1 = 0$

$$y = e^x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(15)(-1)}}{2(15)} = \frac{1}{3} \text{ or } -\frac{1}{5}$$

$$\therefore x = \log_e\left(\frac{1}{3}\right) \text{ or } x = \log_e\left(-\frac{1}{5}\right)$$

since x real $\rightarrow x = \log\left(\frac{1}{3}\right) = -\log 3$.

3. If $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ then prove that $x = a\sqrt{1+b^2} + b\sqrt{1+a^2}$

Soln :- Let $\sinh^{-1} a = \alpha$, $\sinh^{-1} b = \beta$, $\sinh^{-1} x = y$

$$\therefore \alpha + \beta = y$$

$$\sinh(\alpha+\beta) = \sinh(y)$$

$$\sinh\alpha \cosh\beta + \cosh\alpha \sinh\beta = \sinh y \quad \text{---} (I)$$

but $\sinh\alpha = a$, $\sinh\beta = b$, $\sinh y = x$

$$\cosh^2\beta - \sinh^2\beta = 1$$

$$\Rightarrow \cosh\beta = \sqrt{1+\sinh^2\beta} = \sqrt{1+b^2}$$

similarly $\cosh\alpha = \sqrt{1+a^2}$

Substituting in (I)

$$a\sqrt{1+b^2} + b\sqrt{1+a^2} = x$$

4. Prove that $16\, sinh^5 x = \sinh 5x - 5\sinh 3x + 10\sinh x$

Sol^n:- LHS $= 16\sinh^5 x = 16(\sinh x)^5$

$$= 16\left(\frac{e^x - e^{-x}}{2}\right)^5$$

$$= \frac{16}{2^5}\left(e^x - e^{-x}\right)^5$$

$$\left[(a+b)^n = \left(^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_n b^n\right)\right]$$

$$= \frac{16}{2^5}\Big[(e^x)^5 - 5(e^x)^4(e^{-x}) + 10(e^x)^3(e^{-x})^2 - 10(e^x)^2(e^{-x})^3 + 5(e^x)(e^{-x})^4 - (e^{-x})^5\Big]$$

$$= \frac{16}{2^5}\left[e^{5x} - 5e^{3x} + 10e^x - 10e^{-x} + 5e^{-3x} - e^{-5x}\right]$$

$$2^5 \Big[\quad \Big]$$

$$= \frac{1}{2}\left[\left(e^{5x} - e^{-5x}\right) - 5\left(e^{3x} - e^{-3x}\right) + 10\left(e^{x} - e^{-x}\right)\right]$$

$$= \left(\frac{e^{5x} - e^{-5x}}{2}\right) - 5\left(\frac{e^{3x} - e^{-3x}}{2}\right) + 10\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$= \sinh 5x - 5\sinh 3x + 10 \sinh x$$

$= RHS$

5. Prove that $16cosh^5x = \cosh 5x + 5\cosh 3x + 10\cosh x \ \ (HW)$

6. Prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-cosh^2x}}} = cosh^2x$

Soln:- $LHS = \dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\cosh^2 x}}}$

$(\cosh^2 x - \sinh^2 x = 1)$

but $1 - \cosh^2 x = -\sinh^2 x$

$$LHS = \frac{1}{1-\dfrac{1}{1-\dfrac{1}{-\sinh^2 x}}} = \frac{1}{1-\dfrac{1}{1+\operatorname{cosech}^2 x}}$$

$$\left(1 + \operatorname{cosech}^2 x = 1 + \frac{1}{\sinh^2 x} = \frac{\sinh^2 x + 1}{\sinh^2 x} = \frac{\cosh^2 x}{\sinh^2 x} = \coth^2 x\right)$$

$$LHS = \frac{1}{1-\dfrac{1}{\quad}} = \frac{1}{1-\tanh^2 x} = \frac{1}{1-\dfrac{\sinh^2 x}{\cosh^2 x}}$$

$$LHS = \frac{1}{1-\frac{1}{\coth^2 x}} = \frac{}{1-\tanh^2 x} = \frac{}{1-\frac{\sinh^2 x}{\cosh^2 x}}$$

$$= \frac{\cosh^2 x}{\cosh^2 x - \sinh^2 x} = \cosh^2 x = RHS.$$

7. If $u = \log tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, Prove that

(i) $\cosh u = \sec\theta$ **(ii)** $\sinh u = \tan\theta$ **(iii)** $\tanh u = \sin\theta$ **(iv)** $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

Soln :- given $u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$$e^u = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}$$

$$\therefore e^u = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} \qquad \therefore e^{-u} = \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$\therefore (i)\ \cosh u = \frac{e^u+e^{-u}}{2}$$

$$= \frac{1}{2}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)$$

$$= \frac{1}{2}\left[\frac{2(1+\tan^2\theta/2)}{1-\tan^2\frac{\theta}{2}}\right] = \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2}$$

$$\cosh u = \frac{1}{\cos\theta} = \sec\theta.$$

$$(ii)\ \sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta}$$

$$= \tan\theta$$

(iii) $\tanh u = \dfrac{\sinh u}{\cosh u} = \dfrac{\tan\theta}{\sec\theta} = \sin\theta$

(iv) $\tanh\left(\dfrac{u}{2}\right) = \dfrac{\sinh u/2}{\cosh u/2} = \dfrac{2\sinh u/2 \cosh u/2}{2\cosh^2 u/2}$

$$= \frac{\sinh u}{1+\cosh u} = \frac{\tan\theta}{1+\sec\theta} \quad \text{(using (i) \& (ii))}$$

$$\tanh\left(\frac{u}{2}\right) = \frac{\sin\theta/\cos\theta}{1+(1/\cos\theta)} = \frac{\sin\theta}{\cos\theta+1}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\theta/2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\left(\frac{\theta}{2}\right)$$

8. If $\cosh x = \sec\theta$, Prove that

(i) $x = \log(\sec\theta + \tan\theta)$ **(ii)** $\theta = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$ **(iii)** $\tanh\frac{x}{2} = tan\frac{\theta}{2}$

Soln :- $\cosh x = \sec\theta$

$$\frac{e^x + e^{-x}}{2} = \sec\theta$$

$$e^x + e^{-x} = 2\sec\theta$$

$$e^{2x} - 2\sec\theta\, e^x + 1 = 0$$

$e^x = y$ $\quad y^2 - 2\sec\theta\, y + 1 = 0$

$$y = e^x = -(-2\sec\theta) \pm \sqrt{(-2\sec\theta)^2 - 4(1)(1)}$$

$$y = e^x = \frac{-(-2\sec\theta) \pm \sqrt{(-2\sec\theta)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2\sec\theta \pm \sqrt{4\sec^2\theta - 4}}{2}$$

$$= \sec\theta \pm \sqrt{\tan^2\theta}$$

$$e^x = \sec\theta \pm \tan\theta.$$

$$\therefore x = \log(\sec\theta \pm \tan\theta) = \pm\log(\sec\theta + \tan\theta)$$

$\left[\log(\sec\theta - \tan\theta) = -\log(\sec\theta + \tan\theta) \right.$ we can prove this. $\left. \right]$

$$\therefore x = \log(\sec\theta + \tan\theta)$$

(ii) TPT. $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$

let $\tan^{-1}(e^{-x}) = \alpha$

$\therefore e^{-x} = \tan\alpha \qquad \therefore e^x = \cot\alpha$

by the given data $\sec\theta = \cosh x$

$$= \frac{e^x + e^{-x}}{2}$$

$$\therefore \sec\theta = \frac{\tan\alpha + \cot\alpha}{2}$$

$$2\sec\theta = \tan\alpha + \cot\alpha$$

$$= \frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha} = \frac{2}{\sin 2\alpha}$$

$$2\sec\theta = \frac{2}{\sin 2\alpha}$$

$$\therefore \cos\theta = \sin 2\alpha = \cos\left(\frac{\pi}{2} - 2\alpha\right)$$

$$\therefore \theta = \frac{\pi}{2} - 2\alpha = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$$

(iii) TPT $\tanh\left(\frac{x}{2}\right) = \tan\left(\frac{\theta}{2}\right)$

$$\tanh\left(\frac{x}{2}\right) = \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \frac{e^x - 1}{e^x + 1}$$

$$= \frac{\sec\theta + \tan\theta - 1}{\sec\theta + \tan\theta + 1}$$

$$= \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta}$$

$$= \frac{(1 - \cos\theta) + \sin\theta}{(1 + \cos\theta) + \sin\theta}$$

$$= \frac{2\sin^2\theta/2 + 2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2 + 2\sin\theta/2\cos\theta/2}$$

$$= \frac{2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}$$

$$\tanh\left(\frac{x}{2}\right) = \tan\frac{\theta}{2}$$

SEPARATION OF REAL AND IMAGINARY PARTS

Wednesday, October 27, 2021 2:16 PM

Many a time we are required to separate real and imaginary parts of a given complex function. For this, we have to use identities of circular and hyperbolic functions.

In problem where we are given $\tan(\alpha + i\beta) = x + i\,y$, we proceed as shown below

Since $\tan(\alpha + i\beta) = x + i\,y$, we get $\tan(\alpha - i\beta) = x - i\,y$.

$$2\alpha = (\alpha + i\beta) + (\alpha - i\beta)$$

$$\therefore \tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$$

$$\tan(2\alpha) = \tan[\underbrace{(\alpha+i\beta)}_{A} + \underbrace{(\alpha-i\beta)}_{B}]$$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta).\tan(\alpha-i\beta)}$$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-x^2-y^2}$$

$$\therefore 1 - x^2 - y^2 = 2x\cot 2\alpha$$

$$\therefore x^2 + y^2 + 2x\cot 2\alpha - 1 = 0$$

Further, $\tan(2i\beta) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\ \tan(\alpha-i\beta)}$$

$$\tan(\alpha+i\beta) = \frac{\sin(\alpha+i\beta)}{\cos(\alpha+i\beta)}$$

$$i\tanh 2\beta = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \frac{2i\,y}{1+x^2+y^2}$$

$$\therefore \tanh 2\beta = \frac{2y}{1+x^2+y^2}$$

$$\therefore 1 + x^2 + y^2 = 2y\coth 2\beta \qquad \text{i.e.,}\ x^2 + y^2 - 2y\coth 2\beta + 1 = 0$$

SOME SOLVED EXAMPLES:

1. Separate into real and imaginary parts $tan^{-1}(e^{i\,\theta})$

Soln:- Let $\tan^{-1}(e^{i\theta}) = x + iy$

$$\tan(x+iy) = e^{i\theta} = \cos\theta + i\sin\theta$$

$$\therefore \tan(x-iy) = \cos\theta - i\sin\theta$$

$$\tan[(x+iy)+(x-iy)] = \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)}{1-(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}$$

$$\tan(2x) = \frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)} = \frac{2\cos\theta}{0}$$

$$\therefore \tan(2x) = \infty$$

$$\Rightarrow 2x = \frac{\pi}{2} \qquad \therefore \boxed{x = \frac{\pi}{4}}$$

Now $\tan[(x+iy)-(x-iy)]$

$$= \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$\tan(2iy) = \frac{(\cos\theta+i\sin\theta)-(\cos\theta-i\sin\theta)}{1+(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}$$

$$\tan(2iy) = \frac{2i\sin\theta}{1+(\cos^2\theta+\sin^2\theta)} = \frac{2i\sin\theta}{2} = i\sin\theta$$

$(\tan(i\alpha) = i\tanh\alpha)$

$$i\tanh 2y = i\sin\theta$$

$$\therefore \tanh 2y = \sin\theta \Rightarrow 2y = \tanh^{-1}(\sin\theta)$$

$$\therefore y = \frac{1}{2}\tanh^{-1}(\sin\theta)$$

$\therefore \tan^{-1}(e^{i\theta}) = \frac{\pi}{4} + \frac{i}{2}\tanh^{-1}(\sin\theta)$

2. If $\sin(\alpha - i\,\beta) = x + i\,y$ then prove that $\frac{x^2}{cosh^2\beta} + \frac{y^2}{sinh^2\beta} = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = 1$

Soln:- $\sin(\alpha - i\beta) = x + iy$

$\sin\alpha \cos i\beta - \cos\alpha \sin i\beta = x + iy$

$(\cos i\beta = \cosh\beta \quad \& \quad \sin i\beta = i\sinh\beta)$

$\sin\alpha\cosh\beta - i\cos\alpha\sinh\beta = x + iy$

$\Rightarrow x = \sin\alpha\cosh\beta \quad , \quad y = -\cos\alpha\sinh\beta$

(i) $\frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta} = \frac{\sin^2\alpha\cosh^2\beta}{\cosh^2\beta} + \frac{\cos^2\alpha\sinh^2\beta}{\sinh^2\beta}$

$= \sin^2\alpha + \cos^2\alpha = 1.$

(ii) $\frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha} = \frac{\sin^2\alpha\cosh^2\beta}{\sin^2\alpha} - \frac{\cos^2\alpha\sinh^2\beta}{\cos^2\alpha}$

$= \cosh^2\beta - \sinh^2\beta = 1.$

3. If $cos(x + i\,y) = \cos\alpha + i\sin\alpha\,,prove\ that$

(i) $\sin\alpha = \pm sin^2x = \pm \sin h^2 y$ **(ii)** $\cos 2x + \cosh 2y = 2$

Soln:- $\cos(x+iy) = \cos\alpha + i\sin\alpha$

$\Rightarrow \cos x \cos iy - \sin x \sin iy = \cos\alpha + i\sin\alpha$

$\Rightarrow \cos x\cosh y - i\sin x\sinh y = \cos\alpha + i\sin\alpha$

$(\cos iy = \cosh y, \ \sin iy = i\sinh y)$

comparing Real & Imaginary parts

$\cos x \cosh y = \cos\alpha$, $-\sin x \sinh y = \sin\alpha$ — (1)

Now $\cos^2\alpha + \sin^2\alpha = 1$

$\cos^2 x \cdot \cosh^2 y + \sin^2 x \sinh^2 y = 1$

$(1-\sin^2 x)(1+\sinh^2 y) + \sin^2 x \sinh^2 y = 1$

$1 + \sinh^2 y - \sin^2 x - \sin^2 x \sinh^2 y + \sin^2 x \sinh^2 y = 1$

$\sinh^2 y - \sin^2 x = 0$

$\Rightarrow \sin^2 x = \sinh^2 y$ — (2)

$\Rightarrow \sin x = \pm \sinh y$ or $\sinh y = \pm \sin x$

from (1) $\sin\alpha = -\sin x \sinh y = \pm \sin^2 x$

or $\sin\alpha = \pm \sinh^2 y$

(ii) TPT $\cos 2x + \cosh 2y = 2$

LHS $= \cos 2x + \cosh 2y$

$= 1 - 2\sin^2 x + 1 + 2\sinh^2 y$

$= 2 - 2\sin^2 x + 2\sinh^2 y$

but from (2) $\sin^2 x = \sinh^2 y$

$= 2 = $ RHS.

4. If $x + i\,y = \tan(\pi/6 + i\,\alpha)$, prove that $x^2 + y^2 + 2x/\sqrt{3} = 1$

Soln: $\tan\left(\frac{\pi}{6} + i\alpha\right) = x + iy$

$\therefore \tan\left(\frac{\pi}{6} - i\alpha\right) = x - iy$

$$\tan\left[\left(\frac{\pi}{6}+i\alpha\right)+\left(\frac{\pi}{6}-i\alpha\right)\right]$$

$$=\frac{\tan\left(\frac{\pi}{6}+i\alpha\right)+\tan\left(\frac{\pi}{6}-i\alpha\right)}{1-\tan\left(\frac{\pi}{6}+i\alpha\right)\tan\left(\frac{\pi}{6}-i\alpha\right)}=\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}$$

$$\therefore \tan\left(\frac{\pi}{3}\right)=\frac{2x}{1-x^2-y^2}$$

$$\therefore \sqrt{3}=\frac{2x}{1-x^2-y^2} \Rightarrow 1-x^2-y^2=\frac{2}{\sqrt{3}}x$$

$$\Rightarrow x^2+y^2+\frac{2}{\sqrt{3}}x=1$$

5. If $x+i\,y=c\cot(u+i\,v)$, show that $\frac{x}{\sin 2u}=-\frac{y}{\sinh 2v}=\frac{c}{\cosh 2v-\cos 2u}$.

Soln :- $x+iy=c\cot(u+iv)$

$\therefore x-iy=c\cot(u-iv)$

$$\therefore 2x=c\left[\cot(u+iv)+\cot(u-iv)\right]$$

$$=c\left[\frac{\cos(u+iv)}{\sin(u+iv)}+\frac{\cos(u-iv)}{\sin(u-iv)}\right]$$

$$2x=c\left[\frac{\sin(u-iv)\cos(u+iv)+\cos(u-iv)\sin(u+iv)}{\sin(u+iv)\sin(u-iv)}\right]$$

$$\therefore\ 2x = c\left[\frac{\sin[(u-iv)+(u+iv)]}{\frac{1}{2}[\cos(u+iv-u+iv) - \cos(u+iv+u-iv)]}\right]$$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$$\therefore\ 2x = c\left[\frac{\sin 2u}{\frac{1}{2}[\cos 2iv - \cos 2u]}\right]$$

$$\therefore\ x = \frac{c\sin 2u}{\cosh 2v - \cos 2u} \qquad (\cos(2iv) = \cosh 2v)$$

$$\therefore\ \frac{x}{\sin 2u} = \frac{c}{\cosh 2v - \cos 2u}$$

Now,

$$2iy = c\left[\cot(x+iy) - \cot(x-iy)\right]$$

H.W. complete this in similar manner.

6. If $u + i\,v = cosec\left(\frac{\pi}{4} + i\,x\right)$, *prove that* $\left(u^2+v^2\right)^2 = 2\left(u^2 - v^2\right)$

Soln :- $\operatorname{cosec}\left(\frac{\pi}{4} + ix\right) = u+iv$

$$\frac{1}{\sin\left(\frac{\pi}{4}+ix\right)} = u+iv$$

$$\sin\left(\frac{\pi}{4}+ix\right) = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$\sin\frac{\pi}{4}\cos ix + \cos\frac{\pi}{4}\sin ix = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$\sin\frac{\pi}{4}\cos ix + \cos\frac{\pi}{4}\sin ix = \frac{u - iv}{u^2+v^2} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

(Now $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos ix = \cosh x$
$\sin ix = i\sinh x$)

$$\therefore \frac{\cosh x}{\sqrt{2}} + i\frac{\sinh x}{\sqrt{2}} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

comparing both sides

$$\cosh x = \frac{\sqrt{2}u}{u^2+v^2}, \quad \sinh x = \frac{-\sqrt{2}v}{u^2+v^2}$$

Now $\cosh^2 x - \sinh^2 x = 1$

$$\frac{2u^2}{(u^2+v^2)^2} - \frac{2v^2}{(u^2+v^2)^2} = 1$$

$$\therefore 2(u^2-v^2) = (u^2+v^2)^2$$ Hence proved

7. If $x + iy = \cos(\alpha + i\beta)$ or if $\cos^{-1}(x + iy) = \alpha + i\beta$ express x and y in terms of α and β.
Hence show that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation $\lambda^2 - (x^2 + y^2 + 1)\lambda + x^2 = 0$

Soln:- $x+iy = \cos(\alpha+i\beta)$

$= \cos\alpha\cos i\beta - \sin\alpha\sin i\beta$

($\cos i\beta = \cosh\beta$ & $\sin i\beta = i\sinh\beta$)

$x+iy = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

$\therefore x = \cos\alpha\cosh\beta, \quad y = -\sin\alpha\sinh\beta$ —— (1)

wkt, in terms of roots, the quadratic equation is

$$\lambda^2 - (\text{sum of roots})\,\lambda + (\text{product of roots}) = 0$$

To show that $\cos^2\alpha$ & $\cosh^2\beta$ are roots of

$$\lambda^2 - (x^2+y^2+1)\lambda + x^2 = 0$$

it is enough to prove that

$$x^2+y^2+1 = \cos^2\alpha + \cosh^2\beta \quad \text{--- (2)}$$

$$\& \quad x^2 = \cos^2\alpha \cdot \cosh^2\beta \quad \text{--- (3)}$$

from (1) $\quad x = \cos\alpha \cosh\beta$

$$\therefore \quad x^2 = \cos^2\alpha \cosh^2\beta$$

$\therefore$ (3) is proved.

Now $\quad x^2+y^2+1 = \cos^2\alpha\cosh^2\beta + \sin^2\alpha\sinh^2\beta + 1$

$$= \cos^2\alpha\cosh^2\beta + (1-\cos^2\alpha)(\cosh^2\beta - 1) + 1$$

$$= \cos^2\alpha\cosh^2\beta + \cosh^2\beta - 1 - \cos^2\alpha\cosh^2\beta + \cos^2\alpha + 1$$

$$= \cos^2\alpha + \cosh^2\beta$$

$\therefore$ (2) is also proved.

$\therefore \cos^2\alpha$ & $\cosh^2\beta$ are roots of given equation.

INVERSE HYPERBOLIC FUNCTIONS

Friday, October 29, 2021 2:28 PM

If $x = \sinh u$ then $u = \sinh^{-1} x$ is called sine hyperbolic inverse of x, where x is real.
Similarly we can define $\cosh^{-1}x$, $\tanh^{-1}x$, $\coth^{-1}x$, $\text{sech}^{-1}x$, $\text{cosech}^{-1}x$.

Theorem: If x is real.

(i) $\sinh^{-1}x = \log\left(x + \sqrt{x^2+1}\right)$

(ii) $\cosh^{-1}x = \log\left(x + \sqrt{x^2-1}\right)$

(iii) $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Soln :- (i) Let $\sinh^{-1}(x) = y$

$\therefore \sinh y = x$

$$\therefore \frac{e^y - e^{-y}}{2} = x$$

$$\therefore e^y - e^{-y} = 2x$$

multiply by e^y throughout

$$e^{2y} - 1 = 2x e^y$$

$$e^{2y} - 2x e^y - 1 = 0$$

This is a quadratic in e^y

$$\therefore e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$\therefore e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$\therefore y = \log\left(x \pm \sqrt{x^2+1}\right)$$

Now $x - \sqrt{x^2+1} < 0 \qquad \left(x < \sqrt{x^2+1}\right)$

$\therefore \log\left(x - \sqrt{x^2+1}\right)$ is not defined.

$$\therefore \quad y = \log\left(x + \sqrt{x^2+1}\right)$$

$$\therefore \quad \sinh^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$$

(ii) TPT: $\cosh^{-1}(x) = \log\left(x + \sqrt{x^2-1}\right)$

Soln:- Let $\cosh^{-1}(x) = y$

$$\therefore \cosh y = x$$

$$\frac{e^y + e^{-y}}{2} = x$$

$$e^{2y} - 2xe^y + 1 = 0$$

This is a quadratic

$$\therefore e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$e^y = \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

$$e^y = x \pm \sqrt{x^2-1}$$

$$\therefore y = \log\left(x \pm \sqrt{x^2-1}\right) \quad \text{--- (1)}$$

Now $y = \log\left(x - \sqrt{x^2-1}\right)$ --- (2)

$$\therefore e^{y} = x - \sqrt{x^2-1}$$

$$\therefore e^{-y} = \frac{1}{x-\sqrt{x^2-1}} \times \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$$

$$= \frac{x+\sqrt{x^2-1}}{(x)^2 - \left(\sqrt{x^2-1}\right)^2}$$

$$\therefore e^{-y} = x + \sqrt{x^2-1}$$

$$-y = \log\left(x+\sqrt{x^2-1}\right)$$

$$y = -\log\left(x+\sqrt{x^2-1}\right) \quad \text{--- (3)}$$

from (2) & (3) $\log\left(x-\sqrt{x^2-1}\right) = -\log\left(x+\sqrt{x^2-1}\right)$

Subst. in (1) $y = \pm\log\left(x+\sqrt{x^2-1}\right)$

$$\cosh^{-1}x = \pm\log\left(x+\sqrt{x^2-1}\right)$$

$$x = \cosh\left(\pm\log\left(x+\sqrt{x^2-1}\right)\right)$$

[but $\cosh(-z) = \cosh(z)$]

$$x = \cosh\left(\log\left(x+\sqrt{x^2-1}\right)\right)$$

$$\therefore \cosh^{-1} x = \log(x + \sqrt{x^2-1})$$

(iii) TPT: $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

proof :- Let $\tanh^{-1}(x) = y$

$$\therefore x = \tanh y$$

$$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$\frac{1+x}{1-x} = \frac{(e^y + e^{-y}) + (e^y - e^{-y})}{(e^y + e^{-y}) - (e^y - e^{-y})}$$

$$\therefore \frac{1+x}{1-x} = \frac{2e^y}{2e^{-y}} = e^{2y}$$

$$\therefore 2y = \log\left(\frac{1+x}{1-x}\right)$$

$$\therefore y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\therefore \tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

SOME SOLVED EXAMPLES:

1. Prove that $\tanh \log\sqrt{x} = \frac{x-1}{x+1}$ Hence deduce that $\tanh \log\sqrt{5/3} + \tanh \log\sqrt{7} = 1$

Soln :- method 1

$\tanh(y) = \frac{e^y - e^{-y}}{}$

method 2.

Let $\tanh(\log\sqrt{x}) = a$

$$\tanh(y) = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$\tanh(\log\sqrt{x}) = \frac{e^{\log\sqrt{x}} - e^{-\log\sqrt{x}}}{e^{\log\sqrt{x}} + e^{-\log\sqrt{x}}}$$

$$\frac{\sqrt{x} - \frac{1}{\sqrt{x}}}{\sqrt{x} + \frac{1}{\sqrt{x}}}$$

$$= \frac{x-1}{x+1}$$

Let $\tanh(\log\sqrt{x}) = a$

$\therefore \log\sqrt{x} = \tanh^{-1} a$

$$\frac{1}{2}\log x = \frac{1}{2}\log\left(\frac{1+a}{1-a}\right)$$

$$\therefore \frac{x}{1} = \frac{1+a}{1-a}$$

$$\frac{x-1}{x+1} = \frac{(1+a)-(1-a)}{(1+a)+(1-a)}$$

$$\frac{x-1}{x+1} = \frac{2a}{2} = a$$

$$\frac{x-1}{x+1} = \tanh(\log\sqrt{x})$$

$$\tanh(\log\sqrt{x}) = \frac{x-1}{x+1}$$

$$\therefore \tanh\left(\log\sqrt{\frac{5}{3}}\right) = \frac{\frac{5}{3}-1}{\frac{5}{3}+1} = \frac{2}{8}$$

$$\tanh(\log\sqrt{7}) = \frac{7-1}{7+1} = \frac{6}{8}$$

$$\tanh\left(\log\sqrt{\frac{5}{3}}\right) + \tanh(\log\sqrt{7}) = \frac{2}{8} + \frac{6}{8} = 1.$$

2. **(i)** Prove that $cosh^{-1}\sqrt{1+x^2} = sinh^{-1}x$

(ii) Prove that $tanh^{-1}x = sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

(iii) Prove that $cosh^{-1}\left(\sqrt{1+x^2}\right) = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

(iv) Prove that $cot\,h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$ (H.W.) (Proof is similar to $\tanh^{-1}(x)$)

(v) Prove that $sech^{-1}(\sin\theta) = \log cot\frac{\theta}{2}$

~~(iv)~~ Prove that $\cot h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}\log\left(\frac{x+a}{x-a}\right)$ (H.W.) (Proof is similar to $\tanh^{-1}(x)$)

(v) Prove that $sech^{-1}(\sin\theta) = \log\cot\frac{\theta}{2}$

Soln :- (i) let $\cosh^{-1}(\sqrt{1+x^2}) = y$

$$\therefore \sqrt{1+x^2} = \cosh y$$

$$\therefore 1+x^2 = \cosh^2 y$$

$$\therefore x^2 = \cosh^2 y - 1$$

$$\therefore x^2 = \sinh^2 y.$$

$$\therefore x = \sinh y$$

$$\therefore y = \sinh^{-1} x.$$

$$\therefore \cosh^{-1}(\sqrt{1+x^2}) = \sinh^{-1} x$$

(ii) TPT. $\tanh^{-1} x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

Soln :- Let $\tanh^{-1} x = y$

$$\therefore x = \tanh y$$

$$\therefore \frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{1-\tanh^2 y}} = \frac{\tanh y}{\sqrt{\operatorname{sech}^2 y}}$$

$$= \frac{\tanh y}{\operatorname{sech} y} = \sinh y$$

$$\therefore y = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\therefore \tanh^{-1}(x) = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

(iii) Tpt. $\cosh^{-1}\left(\sqrt{1+x^2}\right) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ (H.w.)

Let $\cosh^{-1}\left(\sqrt{1+x^2}\right) = y$

$$\sqrt{1+x^2} = \cosh y$$

(v) Tpt. $\operatorname{sech}^{-1}(\sin\theta) = \log\cot\frac{\theta}{2}$

Soln :- Let $\operatorname{sech}^{-1}(\sin\theta) = x$

$$\sin\theta = \operatorname{sech} x$$

$$\sin\theta = \frac{2}{e^x + e^{-x}}$$

$$\sin\theta = \frac{2e^x}{e^{2x}+1}$$

$$(\sin\theta)\, e^{2x} - 2e^x + \sin\theta = 0$$

This is a quadratic in e^x

$$e^x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(\sin\theta)(\sin\theta)}}{2(\sin\theta)}$$

$$\therefore e^x = \frac{2 \pm \sqrt{4 - 4\sin^2\theta}}{2\sin\theta}$$

$$= \frac{1 \pm \sqrt{1-\sin^2\theta}}{\sin\theta} = \frac{1 \pm \cos\theta}{\sin\theta}$$

$$= \frac{1 \pm \sqrt{1-\sin^2\theta}}{\sin\theta} = \frac{1 \pm \cos\theta}{\sin\theta}$$

$$\therefore e^x = \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2\theta/2}{2\sin\theta/2\,\cos\theta/2}$$

$$e^x = \frac{\cos\theta/2}{\sin\theta/2} = \cot\frac{\theta}{2}$$

$$\therefore x = \log\cot\frac{\theta}{2}$$

$$\therefore \operatorname{sech}^{-1}(\sin\theta) = \log\cot\frac{\theta}{2}$$

3. Separate into real and imaginary parts $\underline{cos^{-1}e^{i\,\theta}}$ ***or*** $\underline{cos^{-1}(\cos\theta + i\sin\theta)}$

Soln :- Let $\cos^{-1} e^{i\theta} = x+iy$

$$\therefore e^{i\theta} = \cos(x+iy)$$

$$\cos\theta + i\sin\theta = \cos x\cos(iy) - \sin x\sin(iy)$$

$$\cos(iy) = \cosh y$$

$$\sin(iy) = i\sinh y$$

$$\therefore \cos\theta + i\sin\theta = \cos x\cosh y - i\sin x\sinh y$$

comparing real & imaginary parts

$$\cos\theta = \cos x\cosh y, \quad \sin\theta = -\sin x\sinh y \quad \text{(1)}$$

Now $\cosh^2 y - \sinh^2 y = 1$

$$\left(\frac{\cos\theta}{\cos x}\right)^2 - \left(\frac{\sin\theta}{-\sin x}\right)^2 = 1$$

$$\frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$$

$$\cdots \quad \sin^2\theta \quad = 1$$

$\cos^2 x \qquad \sin^2 x$

$$\frac{1-\sin^2\theta}{1-\sin^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$$

$$\frac{\sin^2 x(1-\sin^2\theta) - \sin^2\theta(1-\sin^2 x)}{\sin^2 x(1-\sin^2 x)} = 1$$

$$\cancel{\sin^2 x} - \cancel{\sin^2 x \sin^2\theta} - \sin^2\theta + \cancel{\sin^2 x \sin^2\theta} = \cancel{\sin^2 x} - \sin^4 x$$

$$\therefore \sin^2\theta = \sin^4 x$$

$$\therefore \sin x = \sqrt{\sin\theta} \quad \text{——— (2)}$$

$$\therefore x = \sin^{-1}\left(\sqrt{\sin\theta}\right)$$

From (1), $\sin\theta = -\sin x \sinh y$

From (2), $\sin x = \sqrt{\sin\theta}$

$$\therefore \sin\theta = -\sqrt{\sin\theta}\,\sinh y$$

$$\therefore \sinh y = -\sqrt{\sin\theta}$$

$$\therefore y = \sinh^{-1}\left(-\sqrt{\sin\theta}\right)$$

Now $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$

$$\therefore y = \log\left(-\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right)$$

$$\therefore \cos^{-1}(e^{i\theta}) = x + iy = \sin^{-1}\left(\sqrt{\sin\theta}\right) + i\log\left(\sqrt{\sin\theta+1} - \sqrt{\sin\theta}\right)$$

4. Separate into real and imaginary parts $\underline{\sinh^{-1}(i\,x)}$

Soln :- Let $\sinh^{-1}(ix) = \alpha + i\beta$

$$\therefore ix = \sinh(\alpha + i\beta)$$

$\therefore ix = \sinh(\alpha + i\beta)$

$= \sinh\alpha \cosh(i\beta) + \cosh\alpha \sinh(i\beta)$

$\cosh(i\beta) = \cos\beta$

$\sinh(i\beta) = i\sin\beta$

$\therefore ix = \sinh\alpha \cos\beta + i\cosh\alpha \sin\beta$

comparing real & imaginary parts

$\Rightarrow \sinh\alpha \cos\beta = 0$ & $\cosh\alpha \sin\beta = x$ — (1)

Here, $\sinh\alpha \cos\beta = 0$

$\therefore \cos\beta = 0 \Rightarrow \beta = \frac{\pi}{2}$

$\therefore \sin\beta = \sin\frac{\pi}{2} = 1$

$\therefore \cosh\alpha \sin\beta = x \Rightarrow \cosh\alpha = x$

$\Rightarrow \alpha = \cosh^{-1}(x)$

$\sinh\alpha = 0$

$\Rightarrow \alpha = 0$

$\therefore \cosh\alpha = 1$

$\therefore \sin\beta = x \Rightarrow \beta = \sin^{-1}(x)$

$\therefore \sinh^{-1}(ix) = \alpha + i\beta = \cosh^{-1}(x) + i\frac{\pi}{2}$

$\therefore \sinh^{-1}(ix) = \alpha + i\beta = i\sin^{-1}(x)$

5. If $\tan z = \frac{i}{2}(1-i)$, prove that $z = \frac{1}{2}tan^{-1}2 + \frac{i}{4}log\left(\frac{1}{5}\right)$

Soln: $\tan z = \frac{i}{2}(1-i) = \frac{1}{2}(i - i^2) = \frac{1}{2} + \frac{1}{2}i$

Let $z = x + iy$

$\therefore \tan(x+iy) = \frac{1}{2} + \frac{i}{2}$ $\quad \therefore \tan(x-iy) = \frac{1}{2} - \frac{i}{2}$

$\tan(2x) = \tan[(x+iy) + (x-iy)]$

$= \tan(x+iy) + \tan(x-iy)$ $\quad \frac{1}{2} + \frac{i}{2} + \frac{1}{2} - \frac{i}{2}$

$$= \frac{\tan(x+iy) + \tan(x-iy)}{1 - \tan(x+iy)\tan(x-iy)} = \frac{\frac{1}{2}+\frac{i}{2} + \frac{1}{2}-\frac{i}{2}}{1 - \left(\frac{1}{2}+\frac{i}{2}\right)\left(\frac{1}{2}-\frac{i}{2}\right)}$$

$$= \frac{1}{1-\left[\frac{1}{4}+\frac{1}{4}\right]} = 2$$

$$\therefore 2x = \tan^{-1}(2) \Rightarrow x = \frac{1}{2}\tan^{-1}(2)$$

Similarly $\tan(2iy) = \tan\left[(x+iy) - (x-iy)\right]$

$$= \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{\left(\frac{1}{2}+\frac{i}{2}\right) - \left(\frac{1}{2}-\frac{i}{2}\right)}{1+\left(\frac{1}{2}+\frac{i}{2}\right)\left(\frac{1}{2}-\frac{i}{2}\right)}$$

$$i\tanh(2y) = \frac{i}{1+\left(\frac{1}{4}+\frac{1}{4}\right)} = \frac{2}{3}i$$

$(\tan(ix) = i\tanh x)$

$$\therefore \tanh(2y) = \frac{2}{3}$$

$$\therefore 2y = \tanh^{-1}\left(\frac{2}{3}\right)$$

$\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$$2y = \frac{1}{2}\log\left(\frac{1+2/3}{1-2/3}\right)$$

$$\therefore y = \frac{1}{4}\log 5$$

$$\therefore z = x+iy = \frac{1}{2}\tan^{-1}(2) + i\frac{1}{4}\log 5$$

6. Show that $tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \frac{i}{2} log\,\frac{x}{a}$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

Soln :- $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \theta$

$$i\left(\frac{x-a}{x+a}\right) = \tan\theta$$

$$= \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$$

$$\frac{x-a}{x+a} = \frac{e^{i\theta} - e^{-i\theta}}{i^2(e^{i\theta} + e^{-i\theta})} = \frac{e^{-i\theta} - e^{i\theta}}{e^{i\theta} + e^{-i\theta}} \quad (\because i^2 = -1)$$

Componendo - dividendo

$$\frac{(x-a) + (x+a)}{(x-a) - (x+a)} = \frac{(e^{-i\theta} - e^{i\theta}) + (e^{i\theta} + e^{-i\theta})}{(e^{-i\theta} - e^{i\theta}) - (e^{i\theta} + e^{-i\theta})}$$

$$\frac{2x}{-2a} = \frac{2e^{-i\theta}}{-2e^{i\theta}}$$

$$\frac{x}{a} = e^{-2i\theta}$$

$$\therefore -2i\theta = \log\left(\frac{x}{a}\right)$$

$$\therefore \theta = \frac{-1}{2i}\log\left(\frac{x}{a}\right)$$

$$= \frac{-i}{2i^2}\log\left(\frac{x}{a}\right)$$

$$\theta_{\cdot} = \frac{i}{2} \log\left(\frac{x}{a}\right)$$

LOGARITHMS OF COMPLEX NUMBERS

Monday, October 11, 2021   12:12 PM

Let $z = x + iy$ and also let $x = r\cos\theta$, $y = r\sin\theta$ so that $r = \sqrt{x^2+y^2}$ and $\theta = tan^{-1}(y/x)$.

Hence, $\log z = \log(r(\cos\theta + i\sin\theta)) = \log(r.e^{i\theta})$

$= \log r + \log e^{i\theta} = \log r + i\theta$

$\therefore \ \log(x+iy) = \log r + i\theta$

$\therefore \ \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\,tan^{-1}\frac{y}{x}$ .... (1)

This is called **principal value** of log (x + iy)

$\log z = \log r + i\theta$

$\text{Log}\, z = \log r + i(2n\pi + \theta)$

The general value of log (x + iy) is denoted by Log (x + iy) and is given by

$\therefore \text{Log}\,(x+iy) = 2n\pi i + \log(x+iy)$

$\therefore \text{Log}\,(x+iy) = 2n\pi i + \frac{1}{2}\log(x^2+y^2) + i\,tan^{-1}\frac{y}{x}$

$\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i(2n\pi + tan^{-1}\frac{y}{x})$ .... (2)

Caution: $\theta = tan^{-1}y/x$ only when x and y are both positive.

In any other case θ is to be determined from $x = r\cos\theta$, $y = r\sin\theta$, $-\pi \le \theta \le \pi$.

SOME SOLVED EXAMPLES:

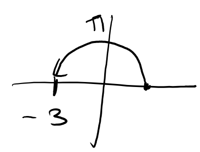
1. Considering the principal value only prove that $\log_2(-3) = \frac{\log 3 + i\pi}{\log 2}$

Sol^n:- $\log_2(-3) = \frac{\log(-3)}{\log 2}$

Now, $\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

$\log(-3) = \frac{1}{2}\log(9+0) + i\tan^{-1}\left(\frac{0}{-3}\right)$

$= \frac{1}{2}\log 9 + i(\pi)$



$\log(-3) = \log 3 + i\pi$

$\therefore \log_2(-3) = \frac{\log 3 + i\pi}{\log 2}$

2. Find the general value of $\text{Log}(1+i) + \text{Log}\,(1-i)$

Sol^n:- $\text{Log}\,(1+i) =$

Sol^n:- $\text{Log}(1+i) =$

$$\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left(2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\text{Log}(1+i) = \frac{1}{2}\log(2) + i(2n\pi + \tan^{-1}(1))$$

$$= \frac{1}{2}\log 2 + i\left(2n\pi + \frac{\pi}{4}\right)$$

$$\text{Log}(1-i) = \frac{1}{2}\log 2 - i\left(2n\pi + \frac{\pi}{4}\right)$$

$$\text{Log}(1+i) + \text{Log}(1-i) = \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = \log 2$$

3. Prove that $\log(1 + e^{2i\theta}) = \log(2\cos\theta) + i\theta$

Sol^n:-

$$\log(1+e^{2i\theta}) = \log(1+\cos 2\theta + i\sin 2\theta)$$

$$= \log(2\cos^2\theta + 2i\sin\theta\cos\theta)$$

$$= \log[2\cos\theta(\cos\theta + i\sin\theta)]$$

$$= \log(2\cos\theta) + \log(\cos\theta + i\sin\theta)$$

$$= \log(2\cos\theta) + \log(e^{i\theta})$$

$$= \log(2\cos\theta) + i\theta$$

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4. Find the value of $\log[\sin(x + iy)]$

Sol^n:

$$\sin(x+iy) = \sin x\cos iy + \cos x\sin iy$$

$$\cos iy = \cosh y$$

$$\sin iy = i\sinh y$$

$$\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$$

$$\log[\sin(x+iy)] = \log[\sin x\cosh y + i\cos x\sinh y]$$

$$\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)$$

$$= \frac{1}{2}\log\left[\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y\right]$$

$$+ i\tan^{-1}\left(\frac{\cos x\sinh y}{\sin x\cosh y}\right) \qquad \text{——— (1)}$$

$$\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$$
$$= (1-\cos^2 x)\cosh^2 y + \cos^2 x(\cosh^2 y - 1)$$
$$= \cosh^2 y - \cos^2 x\cosh^2 y + \cos^2 x\cosh^2 y - \cos^2 x$$
$$= \cosh^2 y - \cos^2 x$$

Sub in (1)

$$\log[\sin(x+iy)] = \frac{1}{2}\log(\cosh^2 y - \cos^2 x) + i\tan^{-1}(\cot x\tanh y)$$

5. Show that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$

Soln:- $$\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\log(a-ib) = \frac{1}{2}\log(a^2+b^2) - i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\log\left(\frac{a-ib}{a+ib}\right) = \log(a-ib) - \log(a+ib)$$

$$\log\left(\frac{a-ib}{a+ib}\right) = -2i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore\ i\log\left(\frac{a-ib}{a+ib}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore \tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan\left(2\tan^{-1}\left(\frac{b}{a}\right)\right)$$

Let $\tan^{-1}\left(\frac{b}{a}\right) = \theta$

$\Rightarrow \frac{b}{a} = \tan\theta$

$$\therefore \tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2(b/a)}{1-(b/a)^2} = \frac{2ab}{a^2-b^2}$$

H.W

6. Prove that $\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{a^2-b^2}{a^2+b^2}$

7. Find the principal value of $(1+i)^{1-i}$

Soln :- Let $z = (1+i)^{1-i}$

Taking log on both sides

$$\log z = (1-i)\log(1+i)$$

$$= (1-i)\left[\frac{1}{2}\log(1^2+1^2) + i\tan^{-1}\left(\frac{1}{1}\right)\right]$$

$$= (1-i)\left[\frac{1}{2}\log(2) + i\left(\frac{\pi}{4}\right)\right]$$

$$\log z = \left[\frac{1}{2}\log 2 + \frac{\pi}{4}\right] + i\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$= x + iy \quad (\text{say})$$

$$z = e^{x+iy} = e^x \cdot e^{iy}$$

$$= e^x\left[\cos y + i\sin y\right]$$

$$z = e^x\cos y + i e^x \sin y$$

real part $= e^{\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right)} \cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)$

real part $= e^{\left(-\frac{1}{2}\log 2+\frac{\pi}{4}\right)} \cos\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)$

Imaginary part $= e^{\left(-\frac{1}{2}\log 2+\frac{\pi}{4}\right)} \sin\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)$

8. Prove that the general value of $(1+i\tan\alpha)^{-i}$ is $e^{2m\pi+\alpha}[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)]$

Soln.: let $z = (1+i\tan\alpha)^{-i}$

Taking general value of Log

$$\log z = (-i)\,\mathrm{Log}\,(1+i\tan\alpha)$$

$$= (-i)\left[\frac{1}{2}\log(1^2+\tan^2\alpha)+i\left(\tan^{-1}\left(\frac{\tan\alpha}{1}\right)+2m\pi\right)\right]$$

$$= (-i)\left[\frac{1}{2}\log(\sec^2\alpha)+i(\alpha+2m\pi)\right]$$

$$= (-i)\left[\log(\sec\alpha)+i(2m\pi+\alpha)\right]$$

$$= (2m\pi+\alpha) - i\log(\sec\alpha)$$

$$\mathrm{Log}\,z = (2m\pi+\alpha)+i\log(\cos\alpha)$$

$$z = e^{(2m\pi+\alpha)}\, e^{i(\log\cos\alpha)}$$

$$z = e^{(2m\pi+\alpha)} \cos(\log(\cos\alpha))+i\sin(\log\cos\alpha)]$$

9. Considering only principal value, if $(1+i\tan\alpha)^{1+i\tan\beta}$ is real, prove that its value is $(\sec\alpha)^{\sec^2\beta}$

Soln.: Let $z = (1+i\tan\alpha)^{1+i\tan\beta}$

$$\log z = (1+i\tan\beta)\log(1+i\tan\alpha)$$

$$= (1+i\tan\beta)\left[\frac{1}{2}\log(1+\tan^2\alpha) + i\tan^{-1}\left(\frac{\tan\alpha}{1}\right)\right]$$

$$= (1+i\tan\beta)\left[\log(\sec\alpha) + i\alpha\right]$$

$$\log z = \left[\log(\sec\alpha) - \alpha\tan\beta\right] + i\left[\alpha + \tan\beta\log(\sec\alpha)\right]$$

$$= x + iy$$

where
$$\left.\begin{aligned} x &= \log(\sec\alpha) - \alpha\tan\beta \\ y &= \alpha + \tan\beta\log(\sec\alpha) \end{aligned}\right\} \quad \text{—— } (1)$$

$$\therefore z = e^{x+iy} = e^{x}\cdot e^{iy} = e^{x}\left[\cos y + i\sin y\right]$$

$$z = e^{x}\cos y + ie^{x}\sin y$$

Since z is real $\Rightarrow e^{x}\sin y = 0$

$$\Rightarrow \sin y = 0 \quad (e^{x} \neq 0)$$

$$\Rightarrow \boxed{y = 0}$$

$$\Rightarrow \alpha + \tan\beta\log(\sec\alpha) = 0 \quad \text{—— } (2)$$

Also $z = e^{x}\cos y + ie^{x}\sin y$

$$= e^{x}\cos(0) = e^{x}$$

$$z = e^{\log(\sec\alpha) - \alpha\tan\beta}$$

$$= e^{\log(\sec\alpha)}\cdot e^{-\alpha\tan\beta}$$

$$= (\sec\alpha)\, e^{-\alpha\tan\beta} \quad (3)$$

$$Z = (\sec\alpha)\, e^{-\alpha\tan\beta} \quad \text{——— } (3)$$

from (2) $\Rightarrow \alpha + \tan\beta \log(\sec\alpha) = 0$

$\Rightarrow -\alpha = \tan\beta \log(\sec\alpha)$

$\Rightarrow -\alpha\tan\beta = \tan^2\beta \log(\sec\alpha)$

$\Rightarrow -\alpha\tan\beta = \log(\sec\alpha)^{\tan^2\beta}$

$\Rightarrow e^{-\alpha\tan\beta} = (\sec\alpha)^{\tan^2\beta}$

Substituting in (3)

$$Z = (\sec\alpha)\, e^{-\alpha\tan\beta} = (\sec\alpha)(\sec\alpha)^{\tan^2\beta}$$

$$= (\sec\alpha)^{1+\tan^2\beta} = (\sec\alpha)^{\sec^2\beta}$$

10. If $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha + i\,\beta,$ find α and β

Soln: Taking log on both sides

$\log(\alpha+i\beta) = (x+iy)\log(a+ib) - (x-iy)\log(a-ib)$

$= (x+iy)\left[\frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)\right]$

$- (x-iy)\left[\frac{1}{2}\log(a^2+b^2) - i\tan^{-1}\left(\frac{b}{a}\right)\right]$

H.W.

11. If $i^{\alpha+i\beta} = \alpha + i\beta$ $\left(\text{or } i^{i^{i\cdots\infty}} = \alpha + i\beta\right)$, prove that $\alpha^2+\beta^2 = e^{-(4n+1)\pi\beta}$ Where n is any positive integer

Soln: $\alpha + i\beta = i^{\alpha+i\beta}$

Sol^n:- $\alpha + i\beta = i^{\alpha + i\beta}$

Taking general value of log

$$\log(\alpha+i\beta) = (\alpha+i\beta)\log(i)$$

$$= (\alpha+i\beta)\log\left[e^{i\left(2n\pi+\frac{\pi}{2}\right)}\right]$$

$$= (\alpha+i\beta)\left[i\left(2n\pi+\frac{\pi}{2}\right)\right]$$

$$\log(\alpha+i\beta) = -\beta\left(2n\pi+\frac{\pi}{2}\right) + i\left(2n\pi+\frac{\pi}{2}\right)\alpha$$

$$(\alpha+i\beta) = e^{-\beta\left(2n\pi+\frac{\pi}{2}\right)} \cdot e^{i\left(2n\pi+\frac{\pi}{2}\right)\alpha}$$

$$= e^{-\beta\left(2n\pi+\frac{\pi}{2}\right)}\left[\cos\left(2n\pi+\frac{\pi}{2}\right)\alpha + i\sin\left(2n\pi+\frac{\pi}{2}\right)\alpha\right]$$

$$\therefore \alpha = e^{-\beta\left(2n\pi+\frac{\pi}{2}\right)}\cos\left(2n\pi+\frac{\pi}{2}\right)\alpha$$

$$\beta = e^{-\beta\left(2n\pi+\frac{\pi}{2}\right)}\sin\left(2n\pi+\frac{\pi}{2}\right)\alpha$$

$$\therefore \alpha^2+\beta^2 = e^{-2\beta\left(2n\pi+\frac{\pi}{2}\right)}\left[\cos^2\left(2n\pi+\frac{\pi}{2}\right)\alpha + \sin^2\left(2n\pi+\frac{\pi}{2}\right)\alpha\right]$$

$$= e^{-(4n\pi+\pi)\beta}$$

$$\alpha^2+\beta^2 = e^{-(4n+1)\pi\beta}$$

12. Prove that $\log tan\left(\frac{\pi}{4}+i\frac{x}{2}\right) = i\, tan^{-1}(\sinh x)$.

Sol^n:- $\log\tan\left(\frac{\pi}{4}+i\frac{x}{2}\right)$

$$= \log\left(\tan\left(\frac{\pi}{4}\right)+\tan\left(i\frac{x}{2}\right)\quad\right)$$

$$= \log\left[\frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(i\frac{x}{2}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(i\frac{x}{2}\right)}\right]$$

$$= \log\left[\frac{1 + \tan\left(i\frac{x}{2}\right)}{1 - \tan\left(i\frac{x}{2}\right)}\right]$$

$$\tan(i\alpha) = i\tanh\alpha$$

$$= \log\left[\frac{1 + i\tanh\left(\frac{x}{2}\right)}{1 - i\tanh\left(\frac{x}{2}\right)}\right]$$

$$= \log\left[1 + i\tanh\left(\frac{x}{2}\right)\right] - \log\left[1 - i\tanh\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{2}\log\left[1 + \tanh^2\left(\frac{x}{2}\right)\right] + i\tan^{-1}\left(\tanh\left(\frac{x}{2}\right)\right)$$

$$- \frac{1}{2}\log\left[1 + \tanh^2\left(\frac{x}{2}\right)\right] + i\tan^{-1}\left(\tanh\left(\frac{x}{2}\right)\right)$$

$$= 2i\tan^{-1}\left(\tanh\left(\frac{x}{2}\right)\right)$$

$$2\tan^{-1}\alpha = \tan^{-1}\left(\frac{2\alpha}{1-\alpha^2}\right)$$

$$\text{LHS} = i\tan^{-1}\left(\frac{2\tanh\left(\frac{x}{2}\right)}{1 - \tanh^2\left(\frac{x}{2}\right)}\right)$$

$= i\tan^{-1}(\sinh x)$

$=$ RHS.

Electrical Networks

circuit $\rightarrow$ i $\rightarrow$ current

$\alpha + i\omega \rightarrow \alpha + j\omega$