LOGARITHMS OF COMPLEX NUMBERS:

Let $z = x + iy$ and also let $x = r\cos\theta$, $y = r\sin\theta$ so that $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}(y/x)$.

Hence, $\log z = \log(r(\cos\theta + i\sin\theta)) = \log(\ r.e^{i\theta})$

$$= \log r + \log e^{i\theta} = \log r + i\theta$$

$\therefore \quad \log(x + iy) = \log r + i\theta$

$\therefore \quad \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\, tan^{-1}\frac{y}{x}$ ……………..(1)

This is called **principal value** of log (x + iy)

The general value of log (x + iy) is denoted by Log (x + iy) and is given by

$\therefore \text{Log }(x + iy) = 2n\pi i + \log(x + iy)$

$\therefore \text{Log }(x + iy) = 2n\pi i + \frac{1}{2}\log(x^2 + y^2) + i\, tan^{-1}\frac{y}{x}$

$\text{Log}(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i(2n\pi + tan^{-1}\frac{y}{x})$ …………….. (2)

Caution: $\theta = tan^{-1} y/x$ only when x and y are both positive.

In any other case θ is to be determined from $x = r\cos\theta$, $y = r\sin\theta$, $-\pi \le \theta \le \pi$.

SOME SOLVED EXAMPLES:

1. Considering the principal value only prove that $\log_2(-3) = \frac{\log 3 + i\pi}{\log 2}$

Solution: Since $log(x + iy) = \frac{1}{2}log(x^2 + y^2) + i\, tan^{-1}\frac{y}{x}$

Putting $x = -3, y = 0$

we have $\log(-3) = \frac{1}{2}(9) + i\, tan^{-1}\left(\frac{0}{-3}\right) = \frac{1}{2}\log 3^2 + i\pi = \log 3 + i\pi$

$log_2(-3) = \frac{log_e(-3)}{log_e 2} = \frac{\log 3 + i\pi}{\log 2}$

2. Find the general value of $\text{Log}(1 + i) + \text{Log }(1 - i)$

Solution: $\log(1 + i) = \frac{1}{2}\log 2 + i\,\frac{\pi}{4} = log\sqrt{2} + i\frac{\pi}{4}$

$\therefore \text{Log}(1 + i) = log\sqrt{2} + i\left(2n\pi + \frac{\pi}{4}\right)$ (General value)

Changing the sign of i,

$\text{Log}(1 - i) = log\sqrt{2} - i\left(2n\pi + \frac{\pi}{4}\right)$

By addition, we get $\text{Log}(1 + i) + \text{Log}(1 - i) = 2\, log\sqrt{2} = 2.\frac{1}{2}\log 2 = \log 2$

3. Prove that $\log(1 + e^{2i\theta}) = \log(2\cos\theta) + i\theta$

Solution: $\log(1 + e^{2i\theta}) = \log(1 + \cos 2\theta + i\sin 2\theta)$

$$= \log(2\cos^2\theta + i2\sin\theta\cos\theta)$$

$$= \log\big(2\cos\theta\,(\cos\theta + i\sin\theta)\big)$$
$$= \log\big(2\cos\theta \cdot e^{i\theta}\big)$$
$$= \log(2\cos\theta) + \log\big(e^{i\theta}\big)$$
$$= \log(2\cos\theta) + i\theta$$

4. Find the value of log $[\sin(x + i\,y)]$

Solution: We have, $\sin(x+iy) = \sin x \cosh y + i\cos\, x\, \sin\, hy$

$$\therefore \log\sin(x+iy) = \frac{1}{2}\log(\sin^2 x\cos h^2\, y + \cos^2 x \sin h^2 y) + i\, tan^{-1}\left(\frac{\cos x \sin hy}{\sin x \cos hy}\right)$$

Now, $sin^2 x\cos h^2 y + cos^2 x\, sinh^2 y = (1 - cos^2 x)\cos h^2 y + cos^2 x\,(\cos h^2 y - 1)$

$$= cosh^2 y - cos^2 x$$
$$= \left(\frac{1+\cos h\, 2\, y}{2}\right) - \left(\frac{1+\cos 2x}{2}\right)$$
$$= \frac{1}{2}(\cos h\, 2\, y - \cos 2x)$$

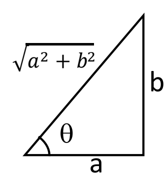
$$\therefore \log\sin(x+iy) = \frac{1}{2}\, log\left(\frac{\cos h\, 2\, y - \cos 2x}{2}\right) + i\, tan^{-1}(\cot x \tan hy)$$

5. Show that $\tan\left[\mathrm{i}\log\left(\frac{\mathrm{a-ib}}{\mathrm{a+ib}}\right)\right] = \frac{2\,\mathrm{ab}}{\mathrm{a^2-b^2}}$

Solution: We have $\log(a - bi) = \frac{1}{2}log\,(a^2+b^2) - itan^{-1}\frac{b}{a}$

And $\log(a+bi) = \frac{1}{2}log(a^2+b^2) + itan^{-1}\frac{b}{a}$

$$\therefore log\left(\frac{a-bi}{a+bi}\right) = \log(a-bi) - \log(a+bi) = -2itan^{-1}\,\frac{b}{a}$$



$$\therefore i\, log\left(\frac{a-bi}{a+bi}\right) = -2i^2 tan^{-1}\frac{b}{a} = 2tan^{-1}\,\frac{b}{a}$$

$$\therefore tan\left\{i\log\left(\frac{a-bi}{a+bi}\right)\right\} = \tan\left(2tan^{-1}\frac{b}{a}\right)$$

$$\therefore tan\left\{i\log\left(\frac{a-bi}{a+bi}\right)\right\} = \tan 2\theta \quad where\ tan^{-1}\frac{b}{a} = \theta$$

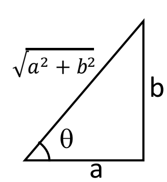
$$= \frac{2\tan\theta}{1-tan^2\theta} = \frac{2(b/a)}{1-(b^2/a^2)} = \frac{2ab}{a^2-b^2}$$

6. Prove that $\cos\left[\mathrm{i}\log\left(\frac{\mathrm{a-ib}}{\mathrm{a+ib}}\right)\right] = \frac{\mathrm{a^2-b^2}}{\mathrm{a^2+b^2}}$

Solution: We have $\log(a - bi) = \frac{1}{2}log\,(a^2+b^2) - itan^{-1}\frac{b}{a}$

And $\log(a+bi) = \frac{1}{2}log(a^2+b^2) + itan^{-1}\frac{b}{a}$

$$\therefore log\left(\frac{a-bi}{a+bi}\right) = \log(a-bi) - \log(a+bi) = -2itan^{-1}\,\frac{b}{a}$$



$$\therefore i\, log\left(\frac{a-bi}{a+bi}\right) = -2i^2 tan^{-1}\frac{b}{a} = 2tan^{-1}\,\frac{b}{a}$$

$$cos\left[i\, log\left(\frac{a-ib}{a+ib}\right)\right] = cos\left(2tan^{-1}\frac{b}{a}\right)$$

$$cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = cos2\theta \quad where\ tan^{-1}\frac{b}{a} = \theta$$

$$= cos^2\theta - sin^2\theta = \frac{a^2}{a^2+b^2} - \frac{b^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$$

7. Separate into real and imaginary parts $\sqrt{i}^{\sqrt{i}}$

Solution: We have $\sqrt{i} = i^{1/2} = \left(cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}$

Also $\sqrt{i} = \left(cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = \left(e^{i\,\pi/2}\right)^{1/2} = e^{i\,\pi/4}$

$\therefore \left(\sqrt{i}\right)^{\sqrt{i}} = \left\{e^{i\,\pi/4}\right\}^{\left(\frac{1}{\sqrt{2}}+i.\frac{1}{\sqrt{2}}\right)} = e^{i\,\pi/4\sqrt{2}-\pi/4\sqrt{2}} = e^{-\pi/4\sqrt{2}}.e^{i\,\pi/4\sqrt{2}}$

$= e^{-\pi/4\sqrt{2}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$

$\therefore$ Real Part $= e^{-\pi/4\sqrt{2}}\cos\left(\frac{\pi}{4\sqrt{2}}\right)$ & Imaginary Part $= e^{-\pi/4\sqrt{2}}\sin\left(\frac{\pi}{4\sqrt{2}}\right)$

8. Find the principal value of $(1+i)^{1-i}$

Solution: $z = (1+i)^{1-i}$

$\therefore \log z = (1-i)\log(1+i)$

$\therefore \log z = (1-i)\left[\log\sqrt{1+1} + \text{i}\tan^{-1}1\right]$

$= (1-i)\left[\frac{1}{2}\log 2 + i.\frac{\pi}{4}\right]$

$= \left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}log2\right) = x + iy\ say$

$\therefore z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y)$

$= e^{\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)}\left[cos\left(\frac{\pi}{4} - \frac{1}{2}log2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}log2\right)\right]$

$= \sqrt{2}e^{\pi/4}\left[cos\left(\frac{\pi}{4} - \frac{1}{2}log2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}log2\right)\right] \qquad \because e^{\frac{1}{2}\log 2} = e^{log\sqrt{2}} = \sqrt{2}$

9. Prove that the general value of $(1 + \text{i}\tan\alpha)^{-\text{i}}$ is $\text{e}^{2\,\text{m}\,\pi+\alpha}[\cos(\log\cos\alpha) + \text{i}\sin(\log\cos\alpha)]$

Solution: Let $1 + i\tan\alpha = r\,e^{\,i\theta}$

$\therefore r^2 = 1 + tan^2\alpha = sec^2\alpha \qquad \therefore r = \sec\alpha$

And $\theta = \tan^{-1}\left(\frac{\tan\alpha}{1}\right) = \tan^{-1}(\tan\alpha) = \alpha$

Now, $Log\ (1 + i\ \tan\alpha) = log\left(r\,e^{\,i\theta}\right) = \log r + (2m\pi + \theta)i$

$= \log\sec\alpha + (2m\pi + \alpha)i$

$\therefore 1 + i\ \tan\alpha = e^{[\log\sec\alpha + (2m\pi+\alpha)i]}$

$\therefore (1 + i\ \tan\alpha)^{-i} = e^{-i[\log\sec\alpha + (2m\pi+\alpha)i]}$

$= e^{2m\pi+\alpha}.e^{-ilog\sec\alpha}$

$= e^{2m\pi+\alpha}.e^{i\,(log\cos\alpha)}$

$= e^{2m\pi+\alpha}.[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)]$

10. Considering only principal value, if $(1+i\tan\alpha)^{1+i\tan\beta}$ is real, prove that its value is $(\sec\alpha)^{sec^2\beta}$

Solution: Let $z=(1+i\ tan\alpha)^{1+i\tan\beta}$

Taking logarithms of both sides,

$$\text{Log}z=(1+i\tan\beta)log(1+i\ tan\alpha)$$

$$=(1+i\tan\beta)\left[\frac{1}{2}log(1+tan^2\alpha)+itan^{-1}tan\alpha\right]$$

$$=(1+i\tan\beta)[\log\sec\alpha+i\alpha]$$

$\therefore \log z=(\log\sec\alpha-\alpha\tan\beta)+i(\alpha+tan\beta\log\sec\alpha)=x+iy$ say

Where $x=\log\sec\alpha-\alpha\tan\beta$ $and\ y=\alpha+tan\beta\log secc\alpha$ …………….(i)

Now, $z=e^{x+iy}=e^x.e^{iy}=e^x(cosy+i\sin y)$

Since by data z is real

$\therefore e^x\sin y=0$ $\quad\therefore y=0$ $\quad\therefore \cos y=1$

$\therefore z=e^x\cos y=e^x=e^{\log sec\alpha-\alpha\tan\beta}$ from (i)

$\therefore z=e^{\log sec\alpha}.e^{-\alpha\tan\beta}=sec\alpha.e^{-\alpha\tan\beta}$ …………………(ii)

But since $y=0$, from (i) $\alpha+tan\beta\log\sec\alpha=0$

$\therefore -\alpha=tan\beta\log sec\alpha$

$\therefore -\alpha\ tan\beta=tan^2\beta.\log sec\ \alpha=log\,(sec\ \alpha)^{tan^2\beta}$

$\therefore e^{-\alpha\tan\beta}=(sec\ \alpha)^{tan^2\beta}$

$\therefore$from (ii) $z=sec\alpha.(sec\ \alpha)^{tan^2\beta}=(sec\ \alpha)^{(1+tan^2\beta)}=(sec\ \alpha)^{sec^2\beta}$

11. If $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}}=\alpha+i\,\beta$, find α and β

Solution: $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}}=\alpha+i\,\beta,$

Taking logarithms of both sides, $log\left(\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}}\right)=log\,(\alpha+i\,\beta)$

$$\log(\alpha+i\beta)=(x+iy)\log(a+ib)-(x-iy)\log(a-ib)$$

$$log(\alpha+i\beta)=(x+iy)\left[\frac{1}{2}log(a^2+b^2)+i\ tan^{-1}\left(\frac{b}{a}\right)\right]-(x-iy)\left[\frac{1}{2}log\,(a^2+b^2)-i\ tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$log(\alpha+i\beta)=2i\left[xtan^{-1}\frac{b}{a}+\frac{y}{2}log(a^2+b^2)\right]$$

$$=2ik\ say \qquad where\ k=\left[xtan^{-1}\frac{b}{a}+\frac{y}{2}log(a^2+b^2)\right]$$

$$\therefore (\alpha+i\beta)=e^{2ik}=cos2k+isin\ 2k$$

$$\therefore \alpha=cos2k,\ \beta=sin\ 2k \qquad where\ k=\left[xtan^{-1}\frac{b}{a}+\frac{y}{2}log(a^2+b^2)\right]$$

12. If $i^{\alpha+i\beta}=\alpha+i\beta$ $\left(\text{or } i^{i^{\cdots\cdots\infty}}=\alpha+i\beta\right)$, prove that $\alpha^2+\beta^2=e^{-(4n+1)\pi\beta}$ Where n is any positive integer

Solution: Since $i=\cos\left(2n\pi+\frac{\pi}{2}\right)+i\sin\left(2n\pi+\frac{\pi}{2}\right)$

we have $i^{\alpha+i\beta}=\alpha+i\beta$

$$\left[\cos\left(2n\pi+\frac{\pi}{2}\right)+\ i\sin\left(2n\pi+\frac{\pi}{2}\right)\right]^{\alpha+i\beta}=\alpha+i\beta$$

$$\therefore e^{i\left(2n\pi+\frac{\pi}{2}\right)(\alpha+i\beta)}=\alpha+i\beta$$

$$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta+i\left(2n\pi+\frac{\pi}{2}\right)\alpha}=\alpha+i\beta$$

$$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}.e^{i\left(2n\pi+\frac{\pi}{2}\right)\alpha}=\alpha+i\beta$$

$$\therefore e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}\left[\cos\left(2n\pi+\frac{\pi}{2}\right)\alpha+i\sin\left(2n\pi+\frac{\pi}{2}\right)\alpha\right]=\alpha+i\beta$$

Equating real and imaginary parts

$e^{-(4n+1)\frac{\pi}{2}\beta}\cos\left(2n\pi+\frac{\pi}{2}\right)\alpha=\alpha$ and $e^{-(4n+1)\frac{\pi}{2}\beta}\sin\left(2n\pi+\frac{\pi}{2}\right)\alpha=\beta$

Squaring and adding, we get $\alpha^2+\beta^2=e^{-(4n+1)\pi\beta}$

13. Prove that $\log tan\left(\frac{\pi}{4}+i\frac{x}{2}\right)=i\ tan^{-1}(\sinh x)$.

Solution: $\log\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)=log\left\{\frac{1+tan(ix/2)}{1-tan(ix/2)}\right\}$

$$=log\left\{\frac{1+\text{itan}\,h\,(x/2)}{1-\text{itan}\,h(x/2)}\right\}$$

$$=log[1+i\tan h\ (x/2)]-log[1-i\tan h(x/2)\ ]$$

$$=\left[\frac{1}{2}log\left(1+tanh^2\left(\frac{x}{2}\right)\right)+itan^{-1}tanh\left(\frac{x}{2}\right)\right]$$

$$-\left[\frac{1}{2}log\left(1+\tan h^2\left(\frac{x}{2}\right)\right)-i\ tan^{-1}\tan h\left(\frac{x}{2}\right)\right]$$

$$=2i\,tan^{-1}\tan h\left(\frac{x}{2}\right)=i.tan^{-1}\left\{\frac{2\tan h(x/2)}{1-tanh^2(x/2)}\right\}=itan^{-1}(\sin hx)$$

$$\because 2tan^{-1}\alpha=tan^{-1}\left\{\frac{2\alpha}{1-\alpha^2}\right\}$$

Practice Problems:

1. Separate into real and imaginary parts

(i) i^i **(ii)** $(-i)^{(i-1)}$ **(iii)** $i^{(i+1)}$

2. If $e^{i\alpha}=i^{\beta}$, prove that $\frac{\alpha}{\beta}=2n\pi+\frac{\pi}{2}$

3. If $(1+i)^{x+iy}=\alpha+i\beta$, prove that $\tan^{-1}\frac{\beta}{\alpha}=\frac{\pi}{4}x+\frac{y}{2}\log 2$

4. Prove that principal value of $(1+i\tan\alpha)^{-i}$ is $e^{\alpha}[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)]$

5. Prove that the real part of the principal value of $(1+i)^{\log i}$ is $e^{-\pi^2/8}\cos\left(\frac{\pi}{4}\log 2\right)$

6. If $\frac{(1+i)^{x+iy}}{(1-i)^{x-iy}} = \alpha + i\,\beta$, find α and β

7. If $\sqrt{i}^{\sqrt{i}\ldots\ldots\infty} = \alpha + i\,\beta$, prove that $\alpha^2 + \beta^2 = e^{-\pi\beta/2}$

8. Find all the root of the equation $tan\,h\,z + 2 = 0$

9. If $\tan[\log(x + i\,y)] = a + i\,b$ Prove that $\tan[log(x^2 + y^2)] = \frac{2a}{1-a^2-b^2}$ when $a^2 + b^2 \neq 1$.

ANSWERS

1. **(i)** $e^{-\left(2n+\frac{1}{2}\right)\pi}$ **(ii)** $e^{\pi/2}(\cos\pi/2 + i\sin\pi/2)$

(iii) $i\,e^{-\pi/2}(\cos\pi/2 + i\sin\pi/2)$

6. $\alpha = \cos k, \beta = \sin k$ where $k = \left(x\frac{\pi}{2} + y\log 2\right)$ **8**. $-\frac{1}{2}\log 3 + i\left(n + \frac{1}{2}\right)\pi$