INVERSE HYPERBOLIC FUNCTIONS:

If $x = \sinh u$ then $u = \sinh^{-1} x$ is called sine hyperbolic inverse of x, where x is real.

Similarly we can define $\cosh^{-1}x$, $\tanh^{-1}x$, $\coth^{-1}x$, $\text{sech}^{-1}x$, $\text{cosech}^{-1}x$.

The inverse hyperbolic functions are many valued but we will consider their principal value only.

Theorem: If x is real.

(i) $\sinh^{-1}x = \log\left(x + \sqrt{x^2+1}\right)$ **(ii)** $\cosh^{-1}x = \log\left(x + \sqrt{x^2-1}\right)$

(iii) $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Proof: (i) $\mathbf{\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)}$

Let $\sinh^{-1} x = y$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - \frac{1}{e^y} = \frac{e^{2y}-1}{e^y}$$

$$e^{2y} - 2x\, e^y - 1 = 0$$

This equation is quadratic in e^y.

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$y = \log\left(x \pm \sqrt{x^2+1}\right)$$

But $x - \sqrt{x^2+1} < 0$ and $\log(-ve)$ is not defined.

$$\therefore y = \log\left(x + \sqrt{x^2+1}\right)$$

$$\sinh^{-1}x = \log\left(x + \sqrt{x^2+1}\right)$$

(ii) $\mathbf{\cosh^{-1}x = \log\left(x + \sqrt{x^2-1}\right)}$

Let $\cosh^{-1}x = y$

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y} = \frac{e^{2y}+1}{e^y}$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$e^y = x \pm \sqrt{x^2-1}$$

$y = \log\left(x \pm \sqrt{x^2-1}\right)$ ……………………….(1)

Consider, $y = \log\left(x - \sqrt{x^2-1}\right)$ …………………..(2)

$$e^y = x - \sqrt{x^2-1},$$

$$e^{-y} = \frac{1}{x-\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \frac{x+\sqrt{x^2-1}}{x^2-x^2+1} = x+\sqrt{x^2-1}$$

$$-y = \log\left(x+\sqrt{x^2-1}\right)$$

$$y = -\log\left(x+\sqrt{x^2-1}\right) \qquad \text{..................(3)}$$

Equating equation (2) and (3), we get

$$\log\left(x-\sqrt{x^2-1}\right) = -\log\left(x+\sqrt{x^2-1}\right) \qquad \text{......................(4)}$$

From equation (1) and (4), we get

$$y = \pm\log\left(x+\sqrt{x^2-1}\right)$$

$$\cosh^{-1}x = \pm\log\left(x+\sqrt{x^2-1}\right)$$

$$\therefore\ x = \cosh\{\pm\log\left(x+\sqrt{x^2-1}\right)\}$$

$$= \cosh\{\log\left(x+\sqrt{x^2-1}\right)\} \qquad \text{since } \cosh(-z) = \cosh z$$

$$\therefore \cosh^{-1}x = \log\left(x+\sqrt{x^2-1}\right)$$

(iii) $\mathbf{\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)}$

Let $\tanh^{-1}x = y$

$$x = \tanh y$$

$$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

Using componendo-dividendo

$$\frac{1+x}{1-x} = \frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y + e^{-y}}$$

$$= \frac{2e^y}{2e^{-y}} = e^{2y}$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \log\left(\frac{1+x}{1-x}\right) \qquad y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

SOME SOLVED EXAMPLES:

1. Prove that $\tanh \log\sqrt{x} = \frac{x-1}{x+1}$ Hence deduce that $\tanh\log\sqrt{5/3} + \tanh\log\sqrt{7} = 1$

Solution: Let $\tanh\log\sqrt{x} = \alpha$

$$\log\sqrt{x} = \tanh^{-1}\alpha$$

$$\frac{1}{2}\log x = \frac{1}{2}\log\left(\frac{1+\alpha}{1-\alpha}\right)$$

$$x = \frac{1+\alpha}{1-\alpha}$$

$$\frac{x-1}{x+1} = \frac{(1+\alpha)-(1-\alpha)}{(1+\alpha)+(1-\alpha)} = \frac{2\alpha}{2} =$$

$\therefore \tan h \log \sqrt{x} = \frac{x-1}{x+1}$

Put $x = 5/3 \quad and \; x = 7$ and add

$\log h(\log\sqrt{5/3}) + \tan h\ (\log\sqrt{7}) = \frac{(5/3)-1}{(5/3)+1} + \frac{7-1}{7+1} = \frac{2}{8} + \frac{6}{8} = 1$

2. **(i)** Prove that $cosh^{-1}\sqrt{1+x^2} = sinh^{-1}x$

Solution: Let $cosh^{-1}\sqrt{1+x^2} = y \quad \therefore \sqrt{1+x^2} = \cos hy$

$\therefore 1+x^2 = \cos h^2 y \quad \therefore x^2 = \cos h^2 y - 1 = \sin h^2 y$

$\therefore x = \sin hy \quad \therefore y = \sin h^{-1}x \quad \therefore \cos h^{-1}\sqrt{1+x^2} = \sin h^{-1}x$

(ii) Prove that $tanh^{-1}x = sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

Solution: Let $\tan h^{-1}x = y \quad \therefore x = \tan hy$

Now, $\frac{x}{\sqrt{1-x^2}} = \frac{\tan hy}{\sqrt{1-tanh^2 y}} = \frac{\tan hy}{\sqrt{cosh^2 y - \sin h^2 y/cosh^2 y}} \quad = \frac{\sin hy}{\cos hy} \times \frac{\cos hy}{1} = \sin hy$

$\therefore y = \sin h^{-1}\frac{x}{\sqrt{1-x^2}} \quad \therefore \tan h^{-1}x = \sin h^{-1}\frac{x}{\sqrt{1-x^2}}$

(iii) Prove that $cosh^{-1}\left(\sqrt{1+x^2}\right) = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

Solution: Let $cosh^{-1}\sqrt{1+x^2} = y \quad \therefore \sqrt{1+x^2} = \cos hy$

$\therefore 1+x^2 = \cos h^2 y \quad \therefore x^2 = \cos h^2 y - 1 = \sin h^2 y \quad \therefore x = \sin hy$

$\therefore \tan hy = \frac{\sin hy}{\cos hy} = \frac{x}{\sqrt{1+x^2}} \quad \therefore y = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\therefore cosh^{-1}\left(\sqrt{1+x^2}\right) = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

(iv) Prove that $cot\ h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$

Solution: Let $cot\ h^{-1}\left(\frac{x}{a}\right) = y \quad \therefore \frac{x}{a} = \cot hy \quad \therefore \tan hy = \frac{1}{\cot hy} = \frac{1}{x/a} = \frac{a}{x}$

$\therefore y = \tan h^{-1}\left(\frac{a}{x}\right) = \frac{1}{2}log\left(\frac{1+(a/x)}{1-(a/x)}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$

$\therefore cot\ h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$

(iii) Prove that $sech^{-1}(\sin\theta) = \log cot\frac{\theta}{2}$

Solution: Let $\sec h^{-1}(\sin\theta) = x \quad \therefore sin\ \theta = \sec hx \quad \therefore \sin\theta = \frac{1}{\cos hx} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x}+1}$

$\therefore (\sin\theta)e^{2x} - 2e^x + \sin\theta = 0$ This is a quadratic in e^x

$\therefore e^x = \frac{2\pm\sqrt{4-4sin^2\theta}}{2\sin\theta} = \frac{1\pm\cos\theta}{\sin\theta}$

$\therefore e^x = \frac{1+\cos\theta}{\sin\theta} = \frac{2\ cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \cot\frac{\theta}{2}$

$\therefore x = \log\cot\left(\frac{\theta}{2}\right)$ $\quad\quad\therefore \sec h^{-1}(\sin\theta) = \log(\cot\theta/2)$

3. Separate into real and imaginary parts $cos^{-1}e^{i\theta}$ ***or*** $cos^{-1}(\cos\theta + i\sin\theta)$

Solution: Let $\cos^{-1} e^{i\theta} = x + iy,\ \ e^{i\theta} = \cos(x+iy)$

$\cos\theta + i\sin\theta = \cos x\cos(iy) - \sin x\sin(iy) = \cos x\cosh y - i\sin x\sinh y$

Equating real and imaginary parts $\cos\theta = \cos x\cosh y$ and $\sin\theta = -\sin x\sinh y$

Since $\cosh^2 y - \sinh^2 y = 1$

$\therefore \left(\frac{\cos\theta}{\cos x}\right)^2 - \left(\frac{-\sin\theta}{\sin x}\right)^2 = 1$

$\therefore \frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$

$\therefore \frac{1-\sin^2\theta}{1-\sin^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$

$\therefore \frac{(1-\sin^2\theta)\sin^2 x - \sin^2\theta(1-\sin^2 x)}{(1-\sin^2 x)\sin^2 x} = 1$

$\therefore \sin^2 x - \sin^2 x\sin^2\theta - \sin^2\theta + \sin^2 x\sin^2\theta = \sin^2 x - \sin^4 x$

$\therefore \sin^2 x - \sin^2\theta = \sin^2 x - \sin^4 x$

$\therefore -\sin^2\theta = -\sin^4 x$

$\therefore \sin^2\theta = \sin^4 x$

$\therefore \sqrt{\sin\theta} = \sin x$ ……………(1)

$\therefore x = \sin^{-1}\sqrt{\sin\theta}$

Since $\sin\theta = -\sin x\sinh y$

$\sin\theta = -\sqrt{\sin\theta}\sinh y$ from (1)

$\therefore -\sqrt{\sin\theta} = \sinh y$

$\therefore y = \sinh^{-1}\left(-\sqrt{\sin\theta}\right) = \log\left(-\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right)$

$\therefore y = \log\left(\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right)$

$\therefore \cos^{-1} e^{i\theta} = x + iy = \sin^{-1}\sqrt{\sin\theta} + i\log\left(\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right)$

4. Separate into real and imaginary parts $sinh^{-1}(i\,x)$

Solution: Let $\sinh^{-1}(ix) = \alpha + i\beta$

$\therefore ix = \sinh(\alpha + i\beta) = \sinh\alpha\cosh(i\beta) + \cosh\alpha\sinh(i\beta)$

$= \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$

Equating real and imaginary parts $\sinh\alpha\cos\beta = 0$

$\therefore \cos\beta = 0 \quad \therefore \beta = \frac{\pi}{2} \quad \therefore \sin\beta = 1$

Also $\cosh\alpha\sin\beta = x$

$\therefore \cosh\alpha = x \quad \left[\because \sin\frac{\pi}{2} = 1\right]$

$\therefore \alpha = \cosh^{-1} x$

$\therefore \sinh^{-1}(ix) = \alpha + i\beta = \cosh^{-1} x + i\frac{\pi}{2}$

5. If $\tan z = \frac{i}{2}(1-i)$, prove that $z = \frac{1}{2}tan^{-1}2 + \frac{i}{4}log\left(\frac{1}{5}\right)$

Solution: $\tan z = \frac{i}{2}(1-i)$

$\tan z = \frac{1}{2}(i - i^2) = \frac{1}{2}i + \frac{1}{2}$

$Let\ z = x + iy \ \ \therefore \tan(x+iy) = \frac{1}{2} + \frac{i}{2}, \quad \tan(x-iy) = \frac{1}{2} - \frac{i}{2}$

$\therefore \tan(2x) = [(x+iy) + (x-iy)]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)} = \frac{\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]+\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]}{1-\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]} = \frac{1}{1-\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\right]} = \frac{1}{1/2} = 2$$

$\therefore 2x = \tan^{-1}2 \qquad \therefore x = \frac{1}{2}\tan^{-1}2$

Now, $tan(2iy) = \tan[(x+iy) - (x-iy)]$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)} = \frac{\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]-\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]}{1+\left[\left(\frac{1}{2}\right)+\left(\frac{i}{2}\right)\right]\left[\left(\frac{1}{2}\right)-\left(\frac{i}{2}\right)\right]} = \frac{i}{1+\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\right]} = \frac{i}{1+(1/2)} = \frac{2}{3}i$$

$\therefore i \tan h\ 2y = \frac{2}{3}i \qquad \therefore \tan h\ 2y = \frac{2}{3}$

$\therefore 2y = tanh^{-1}\left(\frac{2}{3}\right) = \frac{1}{2}log\left[\frac{1+(2/3)}{1-(2/3)}\right] = \frac{1}{2}\log 5 \qquad \therefore y = \frac{1}{4}\log 5$

$\therefore z = x + iy = \frac{1}{2}tan^{-1}2 + \frac{i}{4}\log 5$

6. Show that $tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \frac{i}{2}log\frac{x}{a}$

Solution: Let $tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \theta$

$\therefore i\left(\frac{x-a}{x+a}\right) = \tan\theta = \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$

$\therefore \frac{x-a}{x+a} = \frac{e^{-i\theta}-e^{i\theta}}{e^{i\theta}+e^{-i\theta}} \quad [\because i^2 = -1]$

By componendo and dividend $\frac{(x-a)+(x+a)}{(x-a)-(x+a)} = \frac{(e^{-i\theta}-e^{i\theta})+(e^{i\theta}+e^{-i\theta})}{(e^{-i\theta}-e^{i\theta})-(e^{i\theta}+e^{-i\theta})}$

$\therefore \frac{2x}{-2a} = \frac{2e^{-i\theta}}{-2e^{i\theta}} = e^{-2i\theta} \qquad \therefore \frac{x}{a} = e^{-2i\theta} \qquad \therefore -2i\theta = \log\frac{x}{a}$

Multiply by i throughout, $2\theta = i\log\frac{x}{a} \qquad \therefore \theta = \frac{i}{2}\log\left(\frac{x}{a}\right)$

$tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \frac{i}{2}log\frac{x}{a}$