INVERSE HYPERBOLIC FUNCTIONS

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If $x = \sinh u$ then $u = \sinh^{-1} x$ is called sine hyperbolic inverse of x, where x is real.
Similarly we can define $\cosh^{-1}x, \tanh^{-1}x, \coth^{-1}x, \mathrm{sech}^{-1}x, \mathrm{cosech}^{-1}x$.

Theorem: If x is real.

(i) $\sinh^{-1}x = \log\,(x + \sqrt{x^2+1})$

(ii) $\cosh^{-1}x = \log\,(x + \sqrt{x^2-1})$

(iii) $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Soln :- (i) Let $\sinh^{-1}(x) = y$

$\therefore \sinh y = x$

$$\therefore \frac{e^y - e^{-y}}{2} = x$$

$$\therefore e^y - e^{-y} = 2x$$

multiply by e^y throughout

$$e^{2y} - 1 = 2x e^y$$

$$e^{2y} - 2x e^y - 1 = 0$$

This is a quadratic in e^y

$$\therefore e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$\therefore e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$\therefore y = \log\left(x \pm \sqrt{x^2+1}\right)$$

Now $x - \sqrt{x^2+1} < 0 \quad \left(x < \sqrt{x^2+1}\right)$

$\therefore \log\left(x - \sqrt{x^2+1}\right)$ is not defined.

$$\therefore \quad y = \log\left(x + \sqrt{x^2+1}\right)$$

$$\therefore \quad \sinh^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$$

(ii) TPT $\cosh^{-1}(x) = \log\left(x + \sqrt{x^2-1}\right)$

Soln:- Let $\cosh^{-1}(x) = y$

$$\therefore \cosh y = x$$

$$\frac{e^{y} + e^{-y}}{2} = x$$

$$e^{2y} - 2x e^{y} + 1 = 0$$

This is a quadratic

$$\therefore e^{y} = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$e^{y} = \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

$$e^{y} = x \pm \sqrt{x^2-1}$$

$$\therefore y = \log\left(x \pm \sqrt{x^2-1}\right) \quad \text{—— (1)}$$

Now $\quad y = \log\left(x - \sqrt{x^2-1}\right) \quad$ —— (2)

$$\therefore e^{y} = x - \sqrt{x^2-1}$$

$$\therefore e^{-y} = \frac{1}{x-\sqrt{x^2-1}} \times \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$$

$$= \frac{x+\sqrt{x^2-1}}{(x)^2 - \left(\sqrt{x^2-1}\right)^2}$$

$$\therefore e^{-y} = x + \sqrt{x^2-1}$$

$$-y = \log\left(x+\sqrt{x^2-1}\right)$$

$$y = -\log\left(x+\sqrt{x^2-1}\right) \quad \text{—— (3)}$$

from (2) & (3) $\quad \log\left(x-\sqrt{x^2-1}\right) = -\log\left(x+\sqrt{x^2-1}\right)$

Subst. in (1) $\quad y = \pm\log\left(x+\sqrt{x^2-1}\right)$

$$\cosh^{-1}x = \pm\log\left(x+\sqrt{x^2-1}\right)$$

$$x = \cosh\left(\pm\log\left(x+\sqrt{x^2-1}\right)\right)$$

[but $\cosh(-z) = \cosh(z)$]

$$x = \cosh\left(\log\left(x+\sqrt{x^2-1}\right)\right)$$

$\therefore \cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$

(iii) TPT: $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

proof :- Let $\tanh^{-1}(x) = y$

$$\therefore x = \tanh y$$

$$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$\frac{1+x}{1-x} = \frac{(e^y + e^{-y}) + (e^y - e^{-y})}{(e^y + e^{-y}) - (e^y - e^{-y})}$$

$$\therefore \frac{1+x}{1-x} = \frac{2e^y}{2e^{-y}} = e^{2y}$$

$$\therefore 2y = \log\left(\frac{1+x}{1-x}\right)$$

$$\therefore y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\therefore \tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

SOME SOLVED EXAMPLES:

1. Prove that $\tanh log\sqrt{x} = \frac{x-1}{x+1}$ Hence deduce that $\tanh \log\sqrt{5/3} + \tanh \log\sqrt{7} = 1$

Soln :- method 1

$\tanh(y) = \frac{e^y - e^{-y}}{}$

method 2.

Let $\tanh(\log\sqrt{x}) = a$

$$\tanh(y) = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$\tanh(\log\sqrt{x}) = \frac{e^{\log\sqrt{x}} - e^{-\log\sqrt{x}}}{e^{\log\sqrt{x}} + e^{-\log\sqrt{x}}}$$

$$\frac{\sqrt{x} - \frac{1}{\sqrt{x}}}{\sqrt{x} + \frac{1}{\sqrt{x}}}$$

$$= \frac{x-1}{x+1}$$

Let $\tanh(\log\sqrt{x}) = a$

$\therefore \log\sqrt{x} = \tanh^{-1} a$

$\frac{1}{2}\log x = \frac{1}{2}\log\left(\frac{1+a}{1-a}\right)$

$\therefore \frac{x}{1} = \frac{1+a}{1-a}$

$\frac{x-1}{x+1} = \frac{(1+a)-(1-a)}{(1+a)+(1-a)}$

$\frac{x-1}{x+1} = \frac{2a}{2} = a$

$\frac{x-1}{x+1} = \tanh(\log\sqrt{x})$

$$\tanh(\log\sqrt{x}) = \frac{x-1}{x+1}$$

$$\therefore \tanh\left(\log\sqrt{\frac{5}{3}}\right) = \frac{\frac{5}{3}-1}{\frac{5}{3}+1} = \frac{2}{8}$$

$$\tanh(\log\sqrt{7}) = \frac{7-1}{7+1} = \frac{6}{8}$$

$$\tanh\left(\log\sqrt{\frac{5}{3}}\right) + \tanh(\log\sqrt{7}) = \frac{2}{8} + \frac{6}{8} = 1.$$

2. **(i)** Prove that $cosh^{-1}\sqrt{1+x^2} = sinh^{-1}x$

(ii) Prove that $tanh^{-1}x = sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

(iii) Prove that $cosh^{-1}\left(\sqrt{1+x^2}\right) = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

~~**(iv)**~~ Prove that $cot\,h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$ (H.W.) (Proof is similar to $\tanh^{-1}(x)$)

(v) Prove that $sech^{-1}(\sin\theta) = \log cot\frac{\theta}{2}$

~~(iv)~~ Prove that $\cot h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}\log\left(\frac{x+a}{x-a}\right)$ (H.W.) (Proof is similar to $\tan h^{-1}(x)$)

(v) Prove that $sech^{-1}(\sin\theta) = \log\cot\frac{\theta}{2}$

Soln :- (i) let $\cosh^{-1}(\sqrt{1+x^2}) = y$

$$\therefore \sqrt{1+x^2} = \cosh y$$

$$\therefore 1+x^2 = \cosh^2 y$$

$$\therefore x^2 = \cosh^2 y - 1$$

$$\therefore x^2 = \sinh^2 y.$$

$$\therefore x = \sinh y$$

$$\therefore y = \sinh^{-1} x.$$

$$\therefore \cosh^{-1}(\sqrt{1+x^2}) = \sinh^{-1} x$$

(ii) TPT. $\tanh^{-1} x = \sinh^{-1} \frac{x}{\sqrt{1-x^2}}$

Soln :- Let $\tanh^{-1} x = y$

$$\therefore x = \tanh y$$

$$\therefore \frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{1-\tanh^2 y}} = \frac{\tanh y}{\sqrt{\operatorname{sech}^2 y}}$$

$$= \frac{\tanh y}{\operatorname{sech} y} = \sinh y$$

$$\therefore y = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\therefore \tanh^{-1}(x) = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

(iii) Tpt: $\cosh^{-1}\left(\sqrt{1+x^2}\right) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ (H.W.)

Let $\cosh^{-1}\left(\sqrt{1+x^2}\right) = y$

$$\sqrt{1+x^2} = \cosh y$$

(v) Tpt: $\operatorname{sech}^{-1}(\sin\theta) = \log \cot\frac{\theta}{2}$

Soln :- Let $\operatorname{sech}^{-1}(\sin\theta) = x$

$$\sin\theta = \operatorname{sech} x$$

$$\sin\theta = \frac{2}{e^x + e^{-x}}$$

$$\sin\theta = \frac{2e^x}{e^{2x}+1}$$

$$(\sin\theta)\, e^{2x} - 2e^x + \sin\theta = 0$$

This is a quadratic in e^x

$$e^x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(\sin\theta)(\sin\theta)}}{2(\sin\theta)}$$

$$\therefore e^x = \frac{2 \pm \sqrt{4 - 4\sin^2\theta}}{2\sin\theta}$$

$$= \frac{1 \pm \sqrt{1-\sin^2\theta}}{\sin\theta} = \frac{1 \pm \cos\theta}{\sin\theta}$$

$$= \frac{1 \pm \sqrt{1-\sin^2\theta}}{\sin\theta} = \frac{1 \pm \cos\theta}{\sin\theta}$$

$$\therefore e^x = \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2\theta/2}{2\sin\theta/2 \cos\theta/2}$$

$$e^x = \frac{\cos\theta/2}{\sin\theta/2} = \cot\frac{\theta}{2}$$

$$\therefore x = \log \cot\frac{\theta}{2}$$

$$\therefore \operatorname{sech}^{-1}(\sin\theta) = \log\cot\frac{\theta}{2}$$