Applications of De-Moivre's Theorem

ROOTS OF ALGEBRAIC EQUATIONS:

De Moivre's theorem can be used to find the roots of an algebraic equation.

General values of $\cos\theta = \cos(2k\pi+\theta)$ and $\sin\theta = \sin(2k\pi+\theta)$ where k is an integer.

To solve the equation of the type $z^n = \cos\theta + i\sin\theta$, we apply De Moivre's theorem

$$z = (\cos\theta + i\sin\theta)^{\frac{1}{n}} = \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$$

This shows that $\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ is one of the n roots of $z^n = \cos\theta + i\sin\theta$.

The other roots are obtain by expressing the number in the general form

$$z = \{\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\}^{\frac{1}{n}} = \cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)$$

Taking k = 0, 1, 2,……………,(n – 1). We get n roots of the equation.

Note: (i) Complex roots always occur in conjugate pair if coefficients of different powers of x including constant terms in the equation are real.

(ii) Continued products mean products of all the roots of the equation.

SOME SOLVED EXAMPLES:

1. If ω is a cube root of unity, prove that $(1-\omega)^6 = -27$

Solution: Consider $x^3 = 1$ $\quad\therefore x = 1^{1/3}$

$$\therefore x = (\cos 0 + i\sin 0)^{1/3} = (\cos 2k\pi + i\sin 2k\pi)^{1/3} = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$$

Putting $k = 0, 1, 2$, the cube roots of unity are

$x_0 = 1,\ x_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$ (say)

And $x_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]^2 = \omega^2$

$$\text{Now, } 1+\omega+\omega^2 = 1 + \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - 1 = 0$$

$\therefore 1 + \omega^2 = -\omega$

Now, $(1-\omega)^6 = [(1-\omega)^2]^3 = (1-2\omega+\omega^2)^3$

$$= (-\omega-2\omega)^3 = (-3\omega)^3 - 27\omega^3 = -27$$

2. Find all the values of $\sqrt[3]{(1+i)/\sqrt{2}} + \sqrt[3]{(1-i)/\sqrt{2}}$

Solution: $\sqrt[3]{\frac{(1+i)}{\sqrt{2}}} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{1/3}$

$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3} \qquad = \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{1/3}$$

$$= \left[\cos\left((8k+1)\frac{\pi}{4}\right) + i\sin\left((8k+1)\frac{\pi}{4}\right)\right]^{1/3}$$

$$\sqrt[3]{\frac{(1+i)}{\sqrt{2}}} = \cos\left((8k+1)\frac{\pi}{12}\right) + i\sin\left((8k+1)\frac{\pi}{12}\right)$$

Similarly, $\sqrt[3]{\frac{(1-i)}{\sqrt{2}}} = \cos\left((8k+1)\frac{\pi}{12}\right) - i\sin\left((8k+1)\frac{\pi}{12}\right)$

$$\therefore \sqrt[3]{\frac{(1+i)}{\sqrt{2}}} + \sqrt[3]{\frac{(1-i)}{\sqrt{2}}} = 2\cos\left((8k+1)\frac{\pi}{12}\right)$$

Putting $k = 0, 1, 2$ we get the three roots as

$2\cos\frac{\pi}{12},\ 2\cos\frac{9\pi}{12},\ 2\cos\frac{17\pi}{12}$ i.e., $2\cos\frac{r\pi}{12}$ where $r = 1, 9, 17$

3. Find the cube roots of $(1 - cos\theta - i\ sin\theta)$.

Solution: $(1 - \cos\theta - i\sin\theta)^{1/3} = \left[2\sin^2\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]^{1/3}$

$$= \left[2\sin\left(\frac{\theta}{2}\right)\left(2\sin\left(\frac{\theta}{2}\right) - i\cos\left(\frac{\theta}{2}\right)\right)\right]^{1/3} = \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(2k\pi - \left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right) + i\sin\left(2k\pi - \left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(\frac{(4k-1)+\theta}{6}\right) + i\sin\left(\frac{(4k-1)+\theta}{6}\right)\right]$$

Putting $k = 0, 1, 2$ we get the three roots

4. Find the continued product of all the value of $(-i)^{2/3}$

Solution: $(-i)^{2/3} = \left(0 + i(-1)\right)^{2/3} = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^{2/3}$

$$= \left[\cos\left(2k\pi + \frac{\pi}{2}\right) - i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{2/3}$$

$$= \cos\left((4k+1)\frac{\pi}{3}\right) - i\sin\left((4k+1)\frac{\pi}{3}\right)$$

Putting $k = 0, 1, 2$ we get the three roots as

$$\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right), \left(\cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3}\right), \left(\cos\frac{9\pi}{3} - i\sin\frac{9\pi}{3}\right)$$

$\therefore$ Continued product

$$= \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)\left(\cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3}\right)\left(\cos\frac{9\pi}{3} - i\sin\frac{9\pi}{3}\right)$$

$$= \cos\left(\frac{\pi}{3} + \frac{5\pi}{3} + \frac{9\pi}{3}\right) - i\sin\left(\frac{\pi}{3} + \frac{5\pi}{3} + \frac{9\pi}{3}\right)$$

$$= \cos 5\pi - i\sin 5\pi \qquad = -1 - i(0) \quad = -1$$

5. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1.

Solution: $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}} = \left\{\left(cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3\right\}^{1/4}$

$$= (\cos\pi + i\sin\pi)^{1/4} \qquad = [cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/4}$$

$$= cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}$$

Putting $k = 0,1,2,3$ we get the four roots as,

$$\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \left(cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \left(cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right), \left(cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$\therefore \left(cos\frac{r\pi}{4} + i\ sin\frac{r\pi}{4}\right)$ $where\ r = 1,3,5,7$

The required product $= cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + isin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$

$$= cos4\pi + i\sin 4\pi = 1.$$

6. SOLVE: $x^7 + x^4 + x^3 + 1 = 0$

Solution: $x^7 + x^4 + x^3 + 1 = 0 \quad \therefore x^4(x^3+1) + (x^3+1) = 0$

$\therefore (x^3+1)(x^4+1) = 0 \quad \therefore x^3 = -1, x^4 = -1$

Consider $x^3 = -1$

$$\therefore x = (-1+i0)^{1/3} = (\cos\pi + i\sin\pi)^{1/3} = [\cos(2k+1)\pi - i\sin(2k+1)\pi]^{1/3}$$

$$= \cos(2k+1)\frac{\pi}{3} + i\sin(2k+1)\frac{\pi}{3}$$

Putting $k = 0, 1, 2$ we get the three roots

Similarly from $x^4 = -1$ we get the remaining four roots as

$x = \cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}$ where $k = 0, 1, 2, 3$

7. SOLVE: $x^4 + x^3 + x^2 + x + 1 = 0$

Solution: $x^4 + x^3 + x^2 + x + 1 = 0$

Multiplying the given equation by $x - 1$, we get $(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$

$\therefore$ We have $x^5 - 1 = 0 \quad \therefore x^5 = 1 = \cos 0 + i\sin 0$

$\therefore x^5 = cos(2k\pi) + i\sin(2k\pi)$

$\therefore x = (\cos 2k\pi + i\sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$

Putting $k = 0, 1, 2, 3, 4$, we get the roots of the equation.

$x_0 = \cos 0 + i \sin 0 = 1,$

$x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \quad x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5},$

$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \quad x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

It is clear that 1 is the roots of $x - 1 = 0$

and the remaining roots are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$

i.e., $cos\frac{2\pi}{5} \pm i\, sin\frac{2\pi}{5}, \quad cos\frac{4\pi}{5} \pm i\, sin\frac{4\pi}{5}$

8. SOLVE: $x^4 - x^2 + 1 = 0$

Solution: $x^4 - x^2 + 1 = 0$

Multiplying the given equation by $(x^2 + 1)$, we get, $(x^2+1)(x^4 - x^2 + 1) = 0$

$\therefore (x^2)^3 + (1)^3 = 0 \qquad \therefore x^6 + 1 = 0 \qquad \therefore x^6 = -1$

$\therefore x = (-1 + 0i)^{1/6} = (\cos\pi + i\sin\pi)^{1/6}$

$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/6} = \cos(2k+1)\frac{\pi}{6} + i\sin(2k+1)\frac{\pi}{6}$

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots of equation

$x_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} \qquad x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$

$x_2 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} \qquad x_3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$x_4 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i(-1) = -i \qquad x_5 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

It is clear that i and $-i$ are the roots of $x^2 + 1 = 0$ and the remaining roots x_0, x_2, x_3, x_5 are roots of $x^4 - x^2 + 1 = 0$

9. Find the roots common to $x^4 + 1 = 0 \; and \; x^6 - i = 0.$

Solution: Consider $x^4 + 1 = 0 \qquad \therefore x^4 = -1$

$x = (-1 + i0)^{1/4} = (\cos\pi + i\sin\pi)^{1/4} = [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/4}$

$x = \cos\left((2k+1)\frac{\pi}{4}\right) + i\sin\left((2k+1)\frac{\pi}{4}\right)$

Putting $k = 0, 1, 2, 3$ we get the three roots as

$x_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = 1 \qquad x_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} \qquad x_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = -\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Now consider, $x^6 - i = 0 \qquad \therefore x^6 = i$

$x = (0 + 1i)^{1/6} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/6} = \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{1/6}$

$= \cos\left((4k+1)\frac{\pi}{12}\right) + i\sin\left((4k+1)\frac{\pi}{12}\right)$

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots as

$x_0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$ $x_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

$x_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_3 = \cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}$

$x_4 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$

$x_5 = \cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12} = -\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$\therefore$ common roots are $\pm\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

10. If $(1+x)^6 + x^6 = 0$

show that $x = -\frac{1}{2} - \frac{i}{2}cot\frac{\theta}{2}$ *where* $\theta = (2n+1)\,\pi/6\,, n = 0,1,2,3,4,5.$

Solution: $(1+x)^6 + x^6 = 0$ $\therefore \frac{(1+x)^6}{x^6} = -1$

$\frac{1+x}{x} = (-1)^{1/6} = (\cos\pi + i\sin\pi)^{1/6} = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/6}$

$= \cos\left((2k+1)\frac{\pi}{6}\right) + i\sin\left((2k+1)\frac{\pi}{6}\right)$

$\frac{x+1-x}{x} = \cos\theta + i\sin\theta - 1$

$\frac{1}{x} = (\cos\theta - 1) + i\sin\theta$

$x = \frac{1}{(\cos\theta-1)+i\sin\theta} \times \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)-i\sin\theta} = \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)^2+\sin^2\theta} = \frac{(\cos\theta-1)-i\sin\theta}{2(1-\cos\theta)}$

$= \frac{-2\sin^2(\theta/2)-i2\sin(\theta/2)\cos(\theta/2)}{2(2\sin^2(\theta/2))}$

$= -\frac{1}{2} - \frac{i}{2}\cot\left(\frac{\theta}{2}\right)$ where $\theta = (2k+1)\frac{\pi}{6}$

11. If one root of $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0\ is\ 1+i$, find all other roots.

Solution: The given equation is $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$

Since one of the root is $1+i$

$\therefore$ other root must be $1-i$ (since roots always occurs as complex conjugate pairs)

$\therefore x = 1 \pm i$ are the two roots

$\therefore x - 1 = \pm i$ $\therefore (x-1)^2 = (\pm i)^2$ $\therefore x^2 - 2x + 1 = -1$

$\therefore x^2 - 2x + 2 = 0$

Now we want to find other two remaining roots for that we divide

$x^4 - 6x^3 + 15x^2 - 18x + 10$ by $x^2 - 2x + 2$ and we obtain

$\therefore x^4 - 6x^3 + 15x^2 - 18x + 10 = (x^2 - 2x + 2)(x^2 - 4x + 5)$

$\therefore$ the remaining two roots are the roots of equation $x^2 - 4x + 5 = 0$

$\therefore x = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)} = \frac{4\pm\sqrt{16-20}}{2} = \frac{4\pm 2i}{2} = 2 \pm i$

$\therefore$ The required remaining roots of given equation are $1 - i,\ 2 \pm i$

12. If $\alpha, \alpha^2, \alpha^3, \alpha^4$, are the roots of $x^5 - 1 = 0$, find them & show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5.$

Solution: We have $x^5 = 1 = \cos O + i \sin O$ $\quad\therefore x^5 = cos(2k\pi) + i\sin(2k\pi)$

$\therefore x = (\cos 2k\pi + i \sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$

Putting $k = 0, 1, 2, 3, 4$, we get the five roots as

$x_0 = \cos 0 + i\sin 0 = 1,\qquad x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$

$x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5},\quad x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5},\quad x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5},$

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$, we see that $x_2 = \alpha^2, x_3 = \alpha^3, x_4 = \alpha^4$

$\therefore$ the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$, and hence

$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = \frac{x^5-1}{x-1}$

$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4 + x^3 + x^2 + x + 1$

Putting $x = 1$, we get $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^5) = 5$

13. Solve the equation $z^4 = i(z-1)^4$ and show that the real part of all the roots is $1/2$.

Solution: We have $z^4 = i(z-1)^4$

$\therefore \left(\frac{z}{z-1}\right)^4 = i = cos\frac{\pi}{2} + i\ sin\frac{\pi}{2} = cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$

$\therefore \frac{z}{z-1} = \left[cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{1/4}$

$= cos(4n+1)\frac{\pi}{8} + i\sin(4n+1)\frac{\pi}{8}$

$\therefore \frac{z}{z-1} = \cos\theta + i\sin\theta$ where $\theta = (4n+1)\frac{\pi}{8}$

$\therefore \frac{z}{z-1-z} = \frac{z}{-1} = \frac{\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$ Simplifying as in the above example, we get

$\therefore \frac{z}{-1} = \frac{-sin(\theta/2) + i\cos(\theta/2)}{2\sin(\theta/2)}$

$\therefore -z = -\frac{1}{2} + \frac{i}{2}\ cot\ \frac{\theta}{2}$ $\qquad\therefore z = \frac{1}{2} - \frac{i}{2}\ cot\ \frac{\theta}{2},\quad where\ \theta = (4n+1)\frac{\pi}{8}$

For, $n = 0, 1, 2$, we get three roots, All these roots have the real part $1/2$

14. If ω is a 7th root of unity, prove that

$S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} = 7$ if n is a multiple of 7 and is equal to zero otherwise.

Solution: We have $x = 1^{\frac{1}{7}} = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{7}}$

$$= \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}, \quad where\ \ n = 0, 1, 2, 3, 4, 5, 6$$

Let $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$\therefore \omega^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7 = \cos 2\pi + i\sin 2\pi = 1 \quad \therefore \omega^{7n} = 1^n = 1$

If n is not a multiple of 7, $\therefore \omega^n \neq 1$

Now, $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \ \dots\dots\dots + \omega^{6n} = \frac{1-\omega^{7n}}{1-\omega^n}$ sum of 7 terms of G.P

$$= \frac{1-1}{1-\omega^n} = \frac{0}{1-\omega^n} = 0$$

If n is a multiple of 7, say $n = 7k$

Then, $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n}$

$$= 1 + (\omega^7)^k + (\omega^7)^{2k} + (\omega^7)^{3k} + (\omega^7)^{4k} + (\omega^7)^{5k} + (\omega^7)^{6k}$$

$$= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$$

15. Prove that $\sqrt{1 + sec(\theta/2)} = \left(1 + e^{i\theta}\right)^{-1/2} + \left(1 + e^{-i\theta}\right)^{-1/2}$

Solution: We have to show that $\sqrt{1 + sec(\theta/2)} = \frac{1}{\sqrt{1+e^{i\theta}}} + \frac{1}{\sqrt{1+e^{-i\theta}}}$

Squaring both sides, we get, $1 + sec\frac{\theta}{2} = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$

We shall prove this result

Now, $r.h.s = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$

$$= \frac{1}{1+e^{i\theta}} + \frac{e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{1+e^{-i\theta}+e^{i\theta}+1}}$$

$$= 1 + \frac{2}{\sqrt{2+(e^{i\theta}+e^{-i\theta})}} = 1 + \frac{2}{\sqrt{2+2\cos\theta}}$$

$$= 1 + \frac{2}{\sqrt{2(1+\cos\theta)}} = 1 + \frac{2}{\sqrt{4cos^2(\theta/2)}}$$

$$= 1 + \frac{2}{2cos(\theta/2)} = 1 + sec\frac{\theta}{2} = l.h.s$$

SOME PRACTICE PROBLEMS

1. Find the cube roots of unity. If ω is a complex cube root of unity prove that

(i) $1+\omega+\omega^2=0$ **(ii)** $\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}=0$

2. Prove that the n nth roots of unity are in geometric progression.

3. Show that the sum of the n nth roots of unity is zero.

4. Prove that the product of n nth roots of unity is $(-1)^{n-1}$

5. Find all the values of the following :

(i) $(-1)^{1/5}$ **(ii)** $(-i)^{1/3}$ **(ix)** $\left(1-i\sqrt{3}\right)^{1/4}$

6. Find the continued product of all the values of $\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{3/4}$

7. Find all the value of $(1+i)^{2/3}$ and find the continued product of these values.

8. Solve the equations

(i) $x^9+8x^6+x^3+8=0$ **(ii)** $x^4-x^3+x^2-x+1=0$

(iii) $(x+1)^8+x^8=0$

9. If $(x+1)^6=x^6, show\ that\ x=-\frac{1}{2}-i\ cot\frac{\theta}{2}\ where\ \theta=\frac{2k\pi}{6}, k=0,1,2,3,4,5.$

10. Show that the roots of $(x+1)^7=(x-1)^7$are given by $\pm i\ cot\frac{r\pi}{7}, r=1,2,3.$

11. If $\alpha,\alpha^2,\alpha^3,\ldots\alpha^6$ are the roots of $x^7-1=0$, find them and prove that $(1-\alpha)(1-\alpha^2)\ldots\ldots\ldots\ldots(1-\alpha^6)=7.$

12. Prove that $x^5-1=(x-1)\left(x^2+2x\ cos\frac{\pi}{5}+1\right)\left(x^2+2x\ cos\frac{3\pi}{5}+1\right)=0.$

13. Solve the equation $z^n=(z+1)^n$ and show that the real part of all the roots is $-1/2$.

14. If $a=e^{i\,2\pi/7}$ and $b=a+a^2+a^4,\ c=a^3+a^5+a^6$**.** then prove that b & c are roots of quadratic equation $x^2+x+2=0$.

15. Prove that **(i)** $\sqrt{1-cosce(\theta/2)}=\left(1-e^{i\theta}\right)^{-1/2}-\left(1-e^{-i\theta}\right)^{-1/2}$

(iv) $\sqrt{1-sce(\theta/2)}=\left(1+e^{i\theta}\right)^{-1/2}-\left(1+e^{-i\theta}\right)^{-1/2}$

16. If $1+2i$ is a root of the equation $x^4-3x^3+8x^2-7x+5=0$, find all the other roots.

Answers:

5. **(i)** $-1,\ cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5}$ **(ii)** $i\ ,\ \pm\frac{\sqrt{3}}{2}-\frac{i}{2}$

(iii) $2^{1/4}\left[cos\frac{(6k+5)\pi}{12}+i\ sin\frac{(6k+5)\pi}{12}\right]\ where\ k=0,1,2,3.$

6. 1

7. $2^{1/3}\left(cos\frac{8\pi k+\pi}{6}+i\sin\frac{8\pi k+\pi}{6}\right), k=0,1,2\ ,\ product=2i$

8. **(i)** $\cos(2k+1)\,\pi/6+i\sin(2k+1)\,\pi/6\ , k=0,1,2,3,4,5$

and $2[\cos(k+1)\,\pi/3+i\sin(2k+1)\,\pi/3]\ ,\ k=0,1,2$

(ii) $\cos(2k+1)\,\pi/5+i\sin(2k+1)\,\pi/5\ , k=0,1,2,3,4$

(iii) $x=1/[cos(2k+1)\,\pi/8+i\ sin(2k+1)\,\pi/8\ -1]\ \ here\ k=0,1,2,3,4,5,6,7$

16. $1-2i,\ \ \left(1\pm i\sqrt{3}\right)/2$