DE MOIVRE'S THEOREM

8. If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$, $c = \cos 2\gamma + i\sin 2\gamma$, prove that $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2\cos(\alpha+\beta-\gamma)$

Solution: $\frac{ab}{c} = \frac{(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + \sin 2\beta)}{(\cos 2\gamma + i\sin 2\gamma)}$

$$= \cos\ (2\alpha+2\beta-2\gamma) + i\sin(2\alpha+2\beta-2\gamma)$$

$$= \cos 2(\alpha+\beta-\gamma) + i\sin 2(\alpha+\beta-\gamma)$$

$$\sqrt{\frac{ab}{c}} = [\cos 2(\alpha+\beta-\gamma) + i\sin 2(\alpha+\beta-\gamma)]^{1/2}$$

$$= \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)$$

Similarly, $\sqrt{\frac{c}{ab}} = \cos(\alpha+\beta-\gamma) - i\sin(\alpha+\beta-\gamma)$

By addition we get $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2\cos(\alpha+\beta-\gamma)$

9. If $x - \frac{1}{x} = 2i\sin\theta$, $y - \frac{1}{y} = 2\,i\sin\Phi$, $z - \frac{1}{z} = 2i\sin\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta+\Phi+\psi)$

(ii) $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\,cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Solution: Since $x - \frac{1}{x} = 2i\sin\theta$ $\therefore x^2 - 2ix\sin\theta - 1 = 0$

Solving the quadratic for x, we get,

$$x = \frac{2i\sin\theta \pm \sqrt{4i^2\sin^2\theta - 4(1)(-1)}}{2(1)} = i\sin\theta \pm \sqrt{1-\sin^2\theta} = i\sin\theta \pm \cos\theta$$

consider $x = \cos\theta + i\sin\theta$

Similarly, $= \cos\Phi + i\sin\Phi$, $z = \cos\psi + i\sin\psi$

(i) $xyz = (\cos\theta + i\sin\theta)(\cos\Phi + i\sin\Phi)(\cos\psi + i\sin\psi)$

$$= \cos(\theta+\Phi+\psi) + i\sin(\theta+\Phi+\psi)$$

$\therefore \frac{1}{xyz} = \cos(\theta + \Phi + \psi) - i\sin(\theta + \Phi + \psi)$

Adding we get $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$

(ii) $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} = \frac{(\cos\theta + i\sin\theta)^{1/m}}{(\cos\Phi + i\sin\Phi)^{1/n}} = \frac{\left(\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\right)}{\left(\cos\frac{\Phi}{n} + i\sin\frac{\Phi}{n}\right)} = \cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right) + i\sin\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Similarly, $\frac{\sqrt[n]{y}}{\sqrt[m]{x}} = \cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right) - i\sin\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Adding we get $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

10. If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, Prove that

$\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$.

Solution: We have $cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$

$\therefore (cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$

$\therefore (\cos\alpha + i\sin\alpha) + 2(\cos\beta + i\sin\beta) + 3(\cos\gamma + i\sin\gamma) = 0$

Let $x = cos\,\alpha + i\sin\alpha, y = 2(\cos\beta + i\sin\beta), z = 3(\cos\gamma + i\sin\gamma)$

$\therefore x + y + z = 0$

$\therefore (x + y + z)^3 = 0$

$\therefore x^3 + y^3 + z^3 + 3(x + y + z)(xy + yz + zx) - 3xyz = 0$

$\therefore x^3 + y^3 + z^3 = 3\,xyz$

$\therefore (\cos\alpha + i\sin\alpha)^3 + 2^3(\cos\beta + i\sin\beta)^3 + 3^3(\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha).2.(\cos\beta + i\sin\beta).3.(\cos\gamma + i\sin\gamma)$

$\therefore$ By De Moivre's Theorem,

$(\cos 3\alpha + i\sin 3\alpha) + 8.(\cos 3\beta + i\sin 3\beta) + 27(\cos 3\gamma + i\sin 3\gamma)$

$= 18[cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$

$(\cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma) + i(\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma)$

$= 18[cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$

Equating imaginary parts, we get the required result.

11. If $x_r = cos\frac{\pi}{3^r} + isin\frac{\pi}{3^r}$, prove that **(i)** $x_1x_2x_3 \dots ad.inf. = i$

(ii) $x_0x_1x_2 \dots ad.inf. = -i$

Solution: We have $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$

Putting $r = 0, 1, 2, 3 \dots \dots \dots$ we get $x_0 = \cos\frac{\pi}{3^0} + i\sin\frac{\pi}{3^0} = \cos\pi + i\sin\pi = -1$

$x_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, $x_2 = \cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2} \dots \dots \dots \dots$ and so on

$x_1\ x_2\ x_3 \dots \dots \dots \dots$

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}\right) \dots \dots \dots$$

$$= \cos\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots\dots\right)\pi + i\sin\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots\dots\right)\pi$$

But $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots\dots..\infty = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$

$x_1\ x_2\ x_3 \dots \dots \dots \dots = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$

Also $x_0\ x_1\ x_2\ x_3 \dots \dots \dots \dots = x_0(i) = (-1)(i) = -i$

12. If $(cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta) \dots \dots [cos(2n-1)\theta + i\sin(2n-1)\theta] = 1$ then show that the general value of θ is $\frac{2\,r\pi}{n^2}$

Solution:

L.H.S $= (cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta) \dots \dots [cos(2n-1)\theta + i\sin(2n-1)\theta]$

$$= cos[1 + 3 + \cdots + (2n-1)]\theta + i\sin[1 + 3 + \cdots + (2n-1)]\theta$$

But $1 + 3 + \cdots + (2n-1)$ is an A.P. with first term 1, the number of terms n and common difference 2.

$\therefore$ The Sum, $S_n = \frac{n}{2}[2a + (n-1).d] = \frac{n}{2}[2 + (n-1).2] = n^2$

$\therefore$ L.H.S = $cos(n^2\theta) + i\sin(n^2\theta)$

R.H.S $= 1 = \cos 2\,r\,\pi + i\sin 2\,r\,\pi$ where $r = 0,1,2 \dots \dots$

Equating the two sides, we get $n^2\theta = 2\,r\,\pi$ $\therefore \theta = \frac{2\,r\,\pi}{n^2}$

13. By using De Moivre's Theorem show that

$$\sin\alpha + \sin 2\alpha + \cdots + \sin 5\alpha = \frac{\sin 3\alpha \sin(5\alpha/2)}{sin\alpha/2}$$

Solution: $\frac{1-z^6}{1-z} = 1 + z + z^2 + z^3 + z^4 + z^5$ ....(i)

Let $z = \cos\alpha + i\sin\alpha$, then by De Moivre's theorem, $z^n = \cos n\alpha + i\sin n\,\alpha$

$\therefore 1 + z + z^2 + z^3 + z^4 + z^5 = 1 + (cos\alpha + i\sin\alpha) + (\cos 2\,\alpha + i\sin 2\alpha)$

$+(cos3\alpha + i\sin 3\alpha) + (cos4\alpha + i\sin 4\alpha) + (cos5\alpha + i\sin 5\alpha)$

$= (1 + cos\alpha + \cos 2\,\alpha + cos3\alpha + cos4\alpha + cos5\alpha)$

$+i\,(\sin\alpha + \sin 2\alpha + \sin 3\alpha + \sin\alpha\, 4 + \sin 5\alpha)$ ....(ii)

Now, $\frac{1-z^6}{1-z} = \frac{1-(cos\alpha+i\sin\alpha)^6}{1-(cos\alpha+i\sin\alpha)} = \frac{1-\cos 6\alpha - i\sin 6\alpha}{1-cos\alpha - i\sin\alpha} = \frac{2sin^2 3\alpha - 2i\, sin3\alpha\cos 3\alpha}{2sin^2(\alpha/2) - 2isin(\alpha/2)\cos(\alpha/2)}$

$$= \frac{\sin 3\alpha(\sin 3\alpha - i\cos 3\alpha)\,[sin(\alpha/2) + i\cos(\alpha/2)]}{sin(\alpha/2)[sin(\alpha/2) - icos(\alpha/2)][\sin(\alpha/2) + i\cos(\alpha/2)]}$$

$$= \frac{\sin 3\alpha(\sin 3\alpha - i\cos 3\alpha)[\sin(\alpha/2) - i\cos(\alpha/2)]}{sin(\alpha/2)[\sin^2(\alpha/2) + \cos^2(\alpha/2)]}$$

$$= \frac{\sin 3\alpha}{sin(\alpha/2)}(\sin 3\alpha - i\cos 3\alpha)[\sin(\alpha/2) - i\cos(\alpha/2)]$$

$$= \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(\frac{\pi}{2} - 3\alpha\right) - isin\left(\frac{\pi}{2} - 3\alpha\right)\right] \times \left[cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$$

$$= \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(-\frac{\pi}{2} + 3\alpha\right) + i\sin\left(-\frac{\pi}{2} + 3\alpha\right)\right] \times \left[cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$$

$$\therefore \frac{1-z^6}{1-z} = \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(3\alpha - \frac{\alpha}{2}\right) + i\sin\left(3\alpha - \frac{\alpha}{2}\right)\right]$$

$$= \frac{\sin 3\alpha}{sin(\alpha/2)}\left[cos\left(\frac{5\alpha}{2}\right) + i\,sin\left(\frac{5\alpha}{2}\right)\right]$$ ....(iii)

Using (i) equating real parts, from (ii) and (iii), we get

$$1 + cos\alpha + \cos 2\,\alpha + \cdots . + cos5\alpha = \frac{\sin 3\alpha . cos(5\alpha/2)}{sin(\alpha/2)}$$

And equating imaginary parts, we get

$$\sin\alpha + \sin 2\alpha + \cdots . + \sin 5\alpha = \frac{\sin 3\alpha . sin(5\alpha/2)}{sin(\alpha/2)}$$

SOME PRACTICE PROBLEMS:

1. Simplify

(i) $\frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^6}{(\cos 4\theta + i\sin 4\theta)^7(\cos\theta - i\sin\theta)^8}$ **(ii)** $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^2}{(\cos 4\theta + i\sin 4\theta)^5(\cos 5\theta - i\sin 5\theta)^4}$

2. Prove that

(i) $\frac{(1+i)^8(1-i\sqrt{3})^3}{(1-i)^6(1+i\sqrt{3})^9} = \frac{i}{32}$ **(ii)** $\frac{(1+i\sqrt{3})^9(1-i)^4}{(\sqrt{3}+i)^{12}(1+i)^4} = -\frac{1}{8}$

3. Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{17}}{(\sqrt{3}-i)^{15}}$

4. Express $(1+7i)(2-i)^{-2}$ in the form of $r(\cos\theta + i\sin\theta)$ and prove that the second power is a negative imaginary number and the fourth power is a negative real number.

5. If $x_n + iy_n = (1+i\sqrt{3})^n$, prove that $x_{n-1}y_n - x_n y_{n-1} = 4^{n-1}\sqrt{3}$.

6. Simplify $(1 + cos\,\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n$

7. Prove that $\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$ Hence deduct that

$\left(1 + sin\frac{\pi}{5} + i\,cos\frac{\pi}{5}\right)^5 + i\left(1 + sin\frac{\pi}{5} - i\,cos\frac{\pi}{5}\right)^5 = 0.$

8. If $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $and\ \overline{z}$ is the conjugate of z find the value of $(z)^{15} + (\overline{z})^{15}$.

9. Prove that, if n is a positive integer, then

(i) $(a+ib)^{m/n} + (a-ib)^{m/n} = 2\left(\sqrt{a^2+b^2}\right)^{m/n} cos\left(\frac{m}{n}tan^{-1}\frac{b}{a}\right)$

(ii) $(\sqrt{3}+i)^{120} + (\sqrt{3}-i)^{120} = 2^{121}$

10. If n is a positive integer, prove that $(1+i)^n + (1-i)^n = 2\ 2^{n/2}\cos n\ \pi/4$

Hence, deduce that $(1+i)^{10} + (1-i)^{10} = 0$

11. Prove that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ is equal to -1 if $n = 3k \pm 1$

and 2 if $n = 3k$ where k is an integer.

12. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$,

prove that $\alpha^n + \beta^n = 2^{n+1}cos(n\pi/3)$.

(i) Deduce that $\alpha^{15} + \beta^{15} = -2^{16}$ **(ii)** Deduce that $\alpha^6 + \beta^6 = 128$

13. If α, β are the roots of the equation $z^2 sin^2\theta - z.\sin 2\theta + 1 = 0$, prove that $\alpha^n + \beta^n = 2\cos n\,\theta\, cosec^n\theta$

14. If $a = \cos 3\alpha + i\sin 3\alpha, b = \cos 3\beta + i\sin 3\beta, c = \cos 3\gamma + i\sin 3\gamma$,

prove that $\sqrt[3]{\frac{ab}{c}} + \sqrt[3]{\frac{c}{ab}} = 2cos(\alpha + \beta - \gamma)$

15. If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\emptyset$, $z + \frac{1}{z} = 2\cos\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$ **(ii)** $\sqrt{xyz} + \frac{1}{\sqrt{xyz}} = 2\,cos\left(\frac{\theta+\Phi+\psi}{2}\right)$

(iii) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\Phi)$ **(iv)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\,cos\left(\frac{\theta}{m} - \frac{\emptyset}{n}\right)$

16. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$, prove that $\frac{(b+c)(c+a)(a+b)}{abc} = 8\,cos\frac{(\alpha-\beta)}{2} cos\frac{(\beta-\gamma)}{2} cos\frac{(\gamma-\alpha)}{2}$.

17. If a, b, c are three complex numbers such that $a + b + c = 0$, prove that

(i) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$ and **(ii)** $a^2 + b^2 + c^2 = 0$

18. If $\cos\alpha + \cos\beta + \cos\gamma = 0\ and \sin\alpha + \sin\beta + \sin\gamma = 0$, Prove that

(i) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$, $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$.

(ii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

(iii) $cos(\alpha + \beta) + cos(\beta + \gamma) + cos(\gamma + \alpha) = 0$.

(iv) $\sin(\alpha + \beta) + sin(\beta + \gamma) + sin(\gamma + \alpha) = 0$.

(v) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\,cos(\alpha + \beta + \gamma)$

(vi) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

19. If $a\cos\alpha + b\cos\beta + c\cos\gamma = a\sin\alpha + b\sin\beta + c\sin\gamma = 0$, Prove that $a^3\cos 3\alpha + b^3\cos 3\beta + c^3\cos 3\gamma = 3abc\cos(\alpha + \beta + \gamma)$ and $a^3\sin 3\alpha + b^3\sin 3\beta + c^3\sin 3\gamma = 3\,abc\,sin(\alpha + \beta + \gamma)$

20. If $x_r = cos\left(\frac{2}{3}\right)^r\pi + i\,sin\left(\frac{2}{3}\right)^r\pi$, prove that

(i) $x_1x_2x_3 \ldots \infty = 1$, **(ii)** $x_0x_1x_2 \ldots \infty = -1$

ANSWERS

1. **(i)** $cos\ 12\theta - i \sin 12\theta$ **(ii)** 1

3. 4, $\frac{\pi}{6}$

4. $z = -1 + i = \sqrt{2}(\cos 3\pi/4 + i \sin 3\pi/4)$

$z^4 = \left(\sqrt{2}\right)^4 cos(3\pi + i \sin 3\pi) = -4$

6. $2^{n+1} cos^n \left(\frac{\theta}{2}\right) cos \left(\frac{n\theta}{2}\right)$

8. -2