DE MOIVRE'S THEOREM

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Statement : For any rational number n the value or one of the values of

$$(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\ \sin n\,\theta$$

1. If $z = cos\,\theta + i\sin\theta$ then

$\frac{1}{z} = z^{-1} = (\cos\theta +\ i\,sin\,\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta -\ i\sin\theta$

i.e. $\frac{1}{z} = \cos\theta -\ i\sin\theta$

2. $(\cos\theta - i\sin\theta)^n = \cos\ n\,\theta - i\ \sin n\,\theta$

For, $(\cos\theta - i\sin\theta)^n = \{\cos\ (-\theta) + i\ \sin\ (-\theta)\}^n$

$= \cos(-n\theta) + isin(-n\theta).$

$= \cos n\,\theta - i\sin\ n\ \theta$

Note : Note carefully that ,

(1) $(\sin\theta + i\cos\theta)^n \neq \sin n\,\theta + i\cos n\,\theta$

But $(\sin\theta + i\cos\theta)^n = [cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)]^n$

$= \cos\ n\left(\frac{\pi}{2} - \theta\right) + i\sin n\ \left(\frac{\pi}{2} - \theta\right)$

(2) $(\cos\ \theta + \mathrm{i}\sin\Phi\ )^n \neq \cos\ n\,\theta +\ i\sin n\ \Phi.$

SOME SOLVED EXAMPLES:

1. Simplify $\frac{(\cos 2\theta - i\sin 2\theta)^7(\cos 3\theta + i\sin 3\theta)^5}{(\cos 3\theta + i\sin 3\theta)^{12}(\cos 5\theta - i\sin 5\theta)^7}$

Solution: $cos2\theta - i\sin 2\theta = (cos\theta + i\sin\theta)^{-2}$

$cos3\theta + i\sin 3\theta = (cos\theta + i\sin\theta)^3$

$cos5\theta - i\sin 5\theta = (cos\theta + i\sin\theta)^{-5}$

$\therefore$ Expression $= \frac{(cos\theta + i\sin\theta)^{-14}(cos\theta + i\sin\theta)^{15}}{(cos\theta + i\sin\theta)^{36}(cos\theta + i\sin\theta)^{-35}} = \frac{(cos\theta + i\sin\theta)^1}{(cos\theta + i\sin\theta)^1} = 1$

2. Prove that $\frac{(1+i)^8(\sqrt{3}-i)^4}{(1-i)^4(\sqrt{3}+i)^8} = -\frac{1}{4}$

Solution: $\frac{(1+i)^8(\sqrt{3}-i)^4}{(1-i)^4(\sqrt{3}+i)^8}$

$$(1+i)^8 = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right]^8 = \left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^8 = \left\{\sqrt{2}e^{i\pi/4}\right\}^8 = 2^4.e^{i\,2\pi}$$

$$(1-i)^4 = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\right]^4 = \left[\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)\right]^4 = \left\{\sqrt{2}e^{-i\pi/4}\right\}^4 = 2^2.e^{-i\,\pi}$$

$$\left(\sqrt{3}-i\right)^4 = \left[2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\right]^4 = \left[2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)\right]^4 = \left\{2e^{-i\pi/6}\right\}^4 = 2^4.e^{-i\,2\pi/3}$$

$$\left(\sqrt{3}+i\right)^8 = \left[2\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\right]^8 = \left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^8 = \left\{2e^{i\pi/6}\right\}^8 = 2^8.e^{i\,4\pi/3}$$

Expression $= \frac{(2^4.e^{i\,2\pi}).(2^4.e^{-i\,2\pi/3})}{(2^2.e^{-i\,\pi}).(2^8.e^{i\,4\pi/3})} = \frac{1}{2^2}.\frac{e^{i\,3\pi}}{e^{i\,2\pi}} = \frac{1}{4}e^{i\,\pi} = \frac{1}{4}(\cos\pi + i\sin\pi) = \frac{-1}{4}$

3. Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}}$

Solution: We have $1+i\sqrt{3} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)$

$$\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right) = 2\left(cos\frac{\pi}{6}-isin\frac{\pi}{6}\right)$$

$$\therefore \frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}} = \frac{2^{16}[cos(\pi/3)+i\sin(\pi/3)]^{16}}{2^{17}[cos(\pi/6)-i\sin(\pi/6)]^{17}}$$

$$= \frac{1}{2}\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)^{16}\left(cos\frac{\pi}{6}-isin\frac{\pi}{6}\right)^{-17}$$

$$\therefore \frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}} = \frac{1}{2}\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)^{16}\left[cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right]^{-17}$$

$$= \frac{1}{2}\left(cos\frac{16\pi}{3}+i\sin\frac{16\pi}{3}\right)\left[cos\left(\frac{17\pi}{6}\right)+i\sin\left(\frac{17\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[cos\left(\frac{16}{3}+\frac{17}{6}\right)\pi+i\sin\left(\frac{16}{3}+\frac{17}{6}\right)\pi\right]$$

$$= \frac{1}{2}\left[cos\left(\frac{49}{6}\right)\pi+i\,sin\left(\frac{49}{6}\right)\pi\right]$$

$$= \frac{1}{2}\left[cos\left(8\pi+\frac{\pi}{6}\right)+i\sin\left(8\pi+\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

Hence, the modulus is $\frac{1}{2}$ and principal value of the argument is $\frac{\pi}{6}$

4. Simplify $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n$

Solution: We have $1 = sin^2\alpha + cos^2\alpha = sin^2\alpha - i^2cos^2\alpha$

$$= (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha)$$

$$\therefore 1 + \sin\alpha + i\cos\alpha = (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha) + (\sin\alpha + i\cos\alpha)$$

$$= (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha + 1)$$

$$\therefore \frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha} = \sin\alpha + i\cos\alpha = cos\left(\frac{\pi}{2}-\alpha\right) + i\sin\left(\frac{\pi}{2}-\alpha\right)$$

$$\therefore \left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = \left\{cos\left(\frac{\pi}{2}-\alpha\right) + i\sin\left(\frac{\pi}{2}-\alpha\right)\right\}^n$$

$$= \cos n\left(\frac{\pi}{2}-\alpha\right) + i\sin n\left(\frac{\pi}{2}-\alpha\right)$$

5. If $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ $and\ \overline{z}$ is the conjugate of z prove that $(z)^{10} + (\overline{z})^{10} = 0.$

Solution: $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ $\therefore \bar{z} = \cos\frac{\pi}{4} - isin\frac{\pi}{4}$

$$\therefore (z)^{10} + (\bar{z})^{10} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{10} + \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{10}$$

$$= \left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) + \left(\cos\frac{10\pi}{4} - i\sin\frac{10\pi}{4}\right)$$

$$= 2\cos\frac{10\pi}{4} = 2\cos\left(\frac{5\pi}{2}\right) = 0$$

(ii) $\left(1+i\sqrt{3}\right)^n + \left(1-i\sqrt{3}\right)^n = 2^{n+1}cos(n\pi/3).$

Solution: $1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$1 - i\sqrt{3} = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$\therefore \left(1+i\sqrt{3}\right)^n + \left(1-i\sqrt{3}\right)^n$$

$$= 2^n\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n + 2^n\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^n$$

$$= 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + 2^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$

$$= 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$

$$= 2^n\left(2\cos\frac{n\pi}{3}\right)$$

$$= 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$$

6. If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2.2^{n/2}\cos n\pi/4$, Hence, deduce that $\alpha^8 + \beta^8 = 32$

Solution: The given equation is $x^2 - 2x + 2 = 0$

$$\therefore x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

$$\therefore \alpha = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\beta = 1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$\therefore \alpha^n + \beta^n = \left[\sqrt{2}\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\left(cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^n$$

$$= 2^{n/2}\left(cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + 2^{n/2}\left(cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

$$= 2^{n/2}\left(cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

$$= \left(\sqrt{2}\right)^n\left(2cos\frac{n\pi}{4}\right)$$

$$= 2 \cdot 2^{n/2}cos\frac{n\pi}{4}$$

Putting $n = 8$ $\quad \alpha^8 + \beta^8 = 2 \cdot 2^4\cos 2\pi = 2^5 = 32$

7. If α, β are the roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$, Prove that $\alpha^3 + \beta^3 = 0$ and $\alpha^3 - \beta^3 = 16\,i$

Solution: The given equation is $x^2 - 2\sqrt{3}x + 4 = 0$

$\therefore x = \frac{2\sqrt{3} \pm \sqrt{12-16}}{2} = \sqrt{3} \pm i = 2\left(\frac{\sqrt{3}}{2} \pm i.\frac{1}{2}\right) = 2\left(cos\frac{\pi}{6} \pm i\sin\frac{\pi}{6}\right)$ are the roots

Let $\alpha = 2\left(cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right),\ \beta = 2\left(cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$$\therefore \alpha^3+\beta^3 = 2^3\left(cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^3 + 2^3\left(cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^3$$

$$= 2^3\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}+\cos\frac{\pi}{2}-i\,sin\frac{\pi}{2}\right] = 2^3.2\,cos\frac{\pi}{2} = 0$$

Similarly, $\alpha^3-\beta^3 = 2^3\left(cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^3 + 2^3\left(cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^3$

$$= 2^3\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}-\cos\frac{\pi}{2}+i\,sin\frac{\pi}{2}\right] = 2^3.2\,i\,sin\,\frac{\pi}{2} = 16\,i$$